Geometric Locus activities in a dynamic geometry system. Non-iconic visualization and instrumental genesis

Actividades sobre Lugares Geométricos desarrolladas en un sistema de geometría dinámica. Visualización no icónica y génesis instrumental

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RESUMEN

En este artículo se presenta un estudio sobre la utilización del concepto de Lugar Geométrico en un sistema de geometría dinámica (GeoGebra) en la transición de la Geometría II (geometría proto-axiomática natural) a la Geometría III (geometría completamente axiomática). La investigación se lleva a cabo con 30 estudiantes de matemáticas, futuros profesores, en la universidad española. Se utiliza la teoría de Espacio de Trabajo Geométrico (ETG) como marco teórico de referencia para describir las génesis figurativas e instrumentales involucradas en los procesos de aprendizaje en entornos informáticos. Para el estudio de estas dos génesis se utiliza los conceptos de visualización icónica versus visualización no-icónica, junto a los conceptos de deconstrucción instrumental y dimensional. Los autores identifican tipologías de imágenes y usos de visualización.

PALABRAS CLAVE:

- *Estrategias de resolución de problemas*
- *Visualización*
- *Aprendizaje interactivo*
- *Diagramas*
- *Lugares geométricos*
- *Formación del profesorado*
- *Representaciones visuales*
- *Razonamiento*
- *Geometría*
- *GeoGebra*

ABSTRACT

This paper presents a study on the use of geometric locus using a dynamic geometry system (GeoGebra) in order to pass from Geometry II (a natural proto-axiomatic Geometry) to Geometry III (a complete axiomatic Geometry). The research was conducted with 30 Spanish college prospective mathematics teachers. The Geometrical Working Space theory (GWS) was used as a theoretical framework to describe the figurative and instrumental genesis processes involved in the learning processes in computer environments. To study these two geneses, iconic visualization versus non-iconic visualization, along with instrumental and dimensional deconstruction concepts, was used. The authors identify typologies of images and visualization uses.

KEY WORDS:

- *Problem-solving strategies*
- *Visualization*
- *Interactive learning*
- *Drawing*
- *Diagrams*
- *Loci*
- *Teacher training*
- *Visual representations*
- *Reasoning*
- *Geometry*
- *GeoGebra*

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RESUMO

Neste artigo apresenta-se um estudo sobre o uso de Lugar Geométrico (Locus) usando um sistema de geometria dinâmica (GeoGebra) na passagem de uma Geometria II (proto-axiomática natural) para uma Geometria III (completamente axiomática). A investigação foi realizada com 30 estudantes universitários espanhóis, futuros professores de Matemática. Foi usada a teoria do Espaço de Trabalho Geométrico (ETG) (Space for a Geometric Work, SGW) como marco teórico de referência para descrever as géneses figurativas e instrumentais envolvidas nos processos de aprendizagem em ambientes computacionais. Para estudar estas duas géneses foi usada a visualização icónica vs. a visualização não-icónica juntamente com os conceitos de desconstrução instrumental e dimensional. Os autores identificam tipologias de imagens e usos de visualização.

PALAVRAS CHAVE:

- *Estratégias de resolução de problemas*
- *Visualização*
- *Aprendizagem interativa*
- *Diagramas*
- *Lugares geométricos (Locus)*
- *Formação de professores*
- *Representações visuais*
- *Raciocínio*
- *Geometria*
- *GeoGebra*

RÉSUMÉ

Cet article est centré sur l'étude de l'utilisation de la notion Lieu Géométrique dans un système de géométrie dynamique (GeoGebra) afin de passer de la Géométrie II (géométrie proto-axiomatique naturelle) à la Géométrie III (géométrie axiomatique). Notre recherche concerne un groupe de trente étudiants de licence de mathématiques, futurs enseignants, à l'Université espagnole. Nous appuierons notamment notre analyse sur la notion d'Espace de Travail Géométrique (ETG) et comment les genèses figuratives et instrumentales sont impliqués dans le processus d'apprentissage dans un environnement informatique. Pour étudier ces deux genèses on utilise la visualisation iconique contre visualisation non-iconique joint à la déconstruction instrumentale et dimensionnelle. Les auteurs identifient typologies d'images et usages de la visualisation.

MOTS CLÉS:

- *Stratégies de résolution de problèmes*
- *Visualisation*
- *Activité d'apprentissage interactif*
- *Diagrammes*
- *Lieu Géométrique*
- *Formation des enseignants*
- *Représentations visuelles*
- *Raisonnement*
- *Géométrie*
- *GeoGebra*

1 Introduction

Using dynamic geometry systems (DGS), researchers try to identify how students explore concepts in various representations, and how they form and link images to visualize mathematical concepts. In this paper we study the concept of locus using DGS, for example, GeoGebra[1].

[1] www.geogebra.org.

Concerning the teaching of this concept, there are important aspects to identify, such as: meanings, definitions, visualizations and representations. In previous studies we found out that geometric locus problems create many difficulties when students try to solve them (Gómez-Chacón and Escribano, 2011).

DGS can only find loci of points that have been effectively constructed in their systems, that is, points that parametrically depend on another one ("parametric locus"). In this context some works show the possibilities of the interconnection between dynamic geometry and computer algebra systems, in order to study more general locus (Botana, 2002; Botana, Abánades, Escribano, 2011).

In general, given a geometric configuration, a locus determined by the point *T* is the set of points given by the different positions of *T* when considering all possible instances of the configuration satisfying some property. In particular, given the point *T*, dependent on the point *M*, which is a point on the one-dimensional object *l* (line, circle, ...), the *locus* defined by *T* and *M* is the set of points traced by *T* as *M* moves along *l*. The points *T* and *M* are referred to as the *tracer* point and the *mover* point of the locus, respectively. We must remark the functional nature of "locus" in a DGS. Locus is defined as the image of an object under an application or transformation: the function that transforms the "mover" into the "tracer". The points on the locus depend parametrically on the points of object where the "mover" lives. Under this condition, the DGS is able to build the image points.

Hence, the choice of this subject of study is motivated by these two reasons: the interest of the algebraic-geometric configurations involved in the concept of locus in the DGS, and the difficulties encountered in the resolution of problems by prospective teachers.

We try to make the teaching context of this kind of problems understandable using the framework of Spaces for a Geometric Work (SGW) (see the introduction of this monograph). The resolution of geometric task, for example the locus, implies the setting of an appropriate space for the geometric work. The *appropriate* SGW needs to meet two conditions: it enables the user to solve the problem within the right geometrical paradigm, and it must be well built, in the sense that its various components are organized in a valid way, to take into account the personal SGW of the students.

In particular, using a training situation of homology about geometric locus with prospective teachers we focus on how figural and instrumental geneses are articulated in the SGW using a DGS. Different geneses do not operate separately, they must interact in order to give the geometric work a meaning that the present study aims to highlight, making the articulation existing between this geneses in the personal SGW of the students more explicit and studying the role that GeoGebra plays in the construction of this geometric space, in order to pass from the Geometry II to Geometry III (Paradigmes géométriques, Houdement and Kuzniak, 2006). In this passage we need to pay attention to structure three essential components in Geometry: real and local space with tangible objects, the devices and a theoretical reference from the properties. Geometry II, which can be considered to be a natural proto-axiomatic Geometry, is constructed on a model which is near to reality but is also constructed on axioms. The demonstrations must be developed within this environment. As regards Geometry III, complete axiomatic Geometry, it is possible to disconnect from reality and only count on the system of axioms. This last Geometry is hardly worked on in compulsory schooling, but it remains in the implicit reference frame of teachers who have studied mathematics at university level, where this formal and logical approach is common. So, when specialists are trying to solve geometric problems, they switch repeatedly between paradigms. When we work on GII, a naturally axiomatic geometry and an axiomatic model of reality based on hypothetical-deductive rules (Houdement and Kuzniak, 2006) is not enough to display iconic problems. These problems are not only related to the eyes of students as drawings, but involve a more axiomatic viewpoint to make a dimensional deconstruction. The use of graphical representations is much more complex: the geometry loci teaching requires both conceptual (geometric axiomatic, functional transformations, functional dependencies), non-iconic visualization as well as instrumental genesis, where instrumental deconstruction is crucial. This is the main hypotheses of this work.

Finally, we note that the construction and use of imagery of any kind in mathematical problem-solving constitute a research challenge because of the difficulty of identifying these processes in each individual. The visual imagery used in mathematics is often personal in nature, related not only to conceptual knowledge and belief systems, but also to personal affects (Gómez-Chacón, 2012). This observation has been considered in the analysis of the personal SGW of the students.

2 Theoretical considerations

In geometry, figures are the visual, discursive and heuristic support. Within the SGW framework, cognitive and epistemological levels need to be articulated to ensure a coherent and complete geometric work (Houdement and Kuzniak, 2006). This process supposes some transformations that can possibly be pinpointed through three fundamental geneses:

- A *figural and semiotic* genesis which provides the tangible objects their status of operating mathematical objects.

- An *instrumental genesis* which transforms artifacts into tools within the construction process.
- A *discursive genesis of proof* which gives a meaning to properties used within mathematical reasoning.

In this paper we will examine some key aspects on how both figural and instrumental geneses are involved in the learning process in a computer environment. As it is noticed in the introduction of this volume, the development of the appropriate space of geometric work in a technological context requires an "extended" visualization (Kuzniak & Richard, 2014). The exploration of mathematical objects is supported on figural-semiotic and instrumental genesis. (See Plane 3 (Fig-Ins) associated with figural and instrumental genesis described in the introduction of this monograph). The subject in the process of instrumental genesis constructs schemes of use. These schemes are not only restricted to the physical world, they are related to a whole symbolic system that can be used. A better understanding of the visualization processes must identify which ones are associated with patterns of use, or with structuring information by sign operations, or to a heuristic function that allows the user to anticipate and plan actions and modes of validation. To consider the epistemological point of view and for the study of the subject's activity, we will consider theoretical aspects of visualization and instrumental deconstruction which are described in the following paragraphs.

2.1. *Figural and semiotic genesis*

In this section we will discuss the working definitions that have guided our analysis of the students' work in this genesis.

2.1.1. *Defining Visualization*

In our study, the analysis of the cognitive processes involved in working with (internal and external) representations when reasoning and solving problems requires a holistic definition of the term "visualization". Arcavi's proposal (Arcavi, 2003, p. 217) has consequently been adopted: *"the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings*".

This definition suggests that visual thinking is a way of reasoning consisting of mental transformations of objects that are either constructed in the mind or in some perceived external "reality." Although in this work we will focus on the meaning of visualization as the use of pictures, images and diagrams, which are produced in the graphic register, we agree with Duval (1999)[2] when he notes that visualization can be produced in any register of representation, as it is referred to processes linked to visual perception and then to vision, which is not limited to only one register. For this study, this notion will be considered for categorizing iconic and non-iconic visualization.

2.1.2. *Iconic visualization vs. no-iconic visualization*

It is important to distinguish two opposite ways of cognitive functioning, in which the processes of recognition of the represented objects are radically different in the geometric work: iconic visualization and non-iconic visualization (Duval, 2005). If we consider the complexity of the process of "seeing", "to see" always involves two levels of operations that are different and independent one from other, although they are often merged into the synergy of the act of seeing. These two levels of operations are discriminative recognition of forms and identification of objects corresponding to the recognized forms. The main cognitive problem is to understand how we step from a discriminative recognition of forms to the identification of the objects we see.

In the iconic visualization, recognition of what forms represent is made by the resemblance to the representing (real) object, or by comparison with a prototypical model of forms (a particular figure works as a model, and the other figures are recognized by their degree of resemblance to this model). The figure is independent of the operations that are performed on it.

The non-iconic visualization is a sequence of operations that allows the recognition of geometric properties, due to the impossibility of obtaining certain configurations, or the invariance of the obtained configurations. The figure is a configuration contextually separated from a net or a more complex organization (Duval, 2005, p.14).

We have found out that in geometric learning with DGS there exists a gap between these two different inputs. And this gap is very important, because only non-iconic visualization is relevant for the geometric processes that we want to produce.

2.1.3. *Typologies of images and uses of visualization*

Studying this iconic/non-iconic visualization gap, we have characterized, in the graphic record, different types of images and uses of visualization, that allow us to analyze, in an operative way, productions from the students.

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2 Duval characterizes visualization as "a bi-dimensional organization of relations between some kinds of units. Through visualization, any organization can be synoptically grasped as a configuration" (Duval, 1999:15).

The analysis of these complementary elements, image typology and use of visualization, was conducted by Presmeg (2006) and Guzmán (2002). In Presmeg's approach, images are described both as functional distinctions between types of imagery and as products (concrete imagery ("picture in the mind"), kinesthetic imagery, dynamic imagery, memory images of formula, pattern imagery). In Guzmán they are categorized from the standpoint of conceptualization, the use of visualization as a reference and its role in mathematization, and the heuristic function of images in problem-solving (isomorphic visualization, homeomorphic visualization, analogical visualization and diagrammatic visualization[3]). This final category was the basis adopted in this paper for addressing the handling of tools in problem-solving and research and the precise distinction between the *iconic* and *heuristic function of images* (Duval, 1999) to analyze students' performance (see Annex 1 y 2). The *heuristic function* was found to be related to *visual methods* and cognitive aspects as part of visualization: the effect of basic knowledge, the processes involved in reasoning mediated by geometrical and spatial concepts and the role of imagery based on analogical visualization that connects two domains of experience and helps in the modeling process.

2.2. *On instrumental genesis*

The instrumental approach developed by Rabardel (1995) enables to explore the dual aspect - between design and use - of any technical object (see also Artigue (2002)). A technical object has been designed in order to contribute to the achievement of specific tasks. From this point of view, it embeds knowledge and has characteristics implemented by its designer. But its potentialities of use have to be enacted by a particular user for the purpose of the own task. The artifact and the modalities of its use are elaborated by a particular user. In this perspective, the instrument is the result of a construction and an appropriation by its user for his personal tasks and activities. The process of appropriation and elaboration of an instrument by a user is named instrumental genesis. It requires the elaboration of schemes of use. In the case of the "Locus" tool, we have seen that it is not immediate or simple.

[3] *Isomorphic visualization:* the objects may correspond "exactly" to the representations. *Homeomorphic visualization:* inter-relationships among some of the elements afford an acceptable simulation of the relationships between abstract objects They serve as a guide for the imagination. *Analogical visualization:* the objects at hand are replaced by that are analogously inter-related. Modeling process. *Diagrammatic visualization:* mental objects and their inter-relationships in connection with aspects of interest are merely represented by diagrams that constitute a useful aid to thinking processes. (Guzmán, 2002).

Our study highlights the didactic needs that determine the distinction between the trajectory (trace) and the locus, at an epistemological level of the concept "Locus", which are present in the creation of the tools "Trace" and "Locus" in a DGS (GeoGebra) and determine the resolution of these problems. To use the "Locus" tool requires an operative apprehension for fruitful intuition of the figure. The non-iconic visualization require awareness of the properties that are linked to operations taking place, either to construct a figure or to transform it, where two types of deconstruction are crucial: *instrumental deconstruction* (procedural dimension, defined by Mithalal as the identification of a set of independent figural units, the primitives, and a succession of actions performed through the use of instruments, that are used to reconstruct the object itself, or a graphical representation of the object) and *dimensional deconstruction* (discursive activity on the geometrical properties of the figural units) (Mithalal, 2010).

❸ Training and research methodology

The study was conducted with 30 mathematics prospective teachers at the Spanish University. The qualitative research methodology used consisted of observation during participation in student training and output analysis sessions as well as semi-structured interviews (video-recording). The procedure used in data collection was student problem solving, along with a questionnaire about visual reasoning and technology difficulties. All screen and audio activity on the students' computers was recorded with CamStudio software, with which video-based information on problem solving with GeoGebra could be generated. Consequently, at least four data sources were available for each student.

Six non-routine geometric locus problems were posed, to be solved using GeoGebra during the training session. The problems are posed in an analytic register. Each problem admits several kinds of resolution, including a visual one. Thus the problems allow the study of the students' behavior with the coordination of registers and they enable us to compare results between those who make a conversion to the graphic register and those who do not. Finding the solutions to the problems required to follow a sequence of visualization, technical, deductive and analytical steps.

TABLE 1
Geometric locus problems

PROBLEM
Problem 1: find the equation for the locus formed by the barycenter of a triangle ABC, where A = (0, 4), B = (4, 0) and C is a point on circle $x^2 + y^2 + 4x = 0$.
Problem 2: assume a variable line r that cuts through the origin O. Take point P to be the point where line r intersects with line Y = 3. Draw line AP from point A = (3,0), and the line perpendicular to AP, s. Find the locus of the intersection points Q between lines r and s, when r is shifted.
Problem 3: assume a triangle ABC and a point P. Project P on the sides of the triangle: Q1, Q2, Q3. Are Q1, Q2 and Q3 on the same line? Define the locus for points P when Q1, Q2 and Q3 are aligned.
Problem 4: the top of a 5-meter ladder rests against a vertical wall, and the bottom on the ground. Define the locus generated by midpoint M of the ladder when it slips and falls to the ground. Define the locus for any other point on the ladder.
Problem 5: find the locus of points such that the ratio of their distances to points A = (2, -3) and B = (3, -2) is 5/3. Identify the geometric object formed.
Problem 6: find the equation for the locus of point P such that the sum of the distances to the axes equals the square of the distance to the origin. Identify the geometric object formed.

Geometric locus training was conducted in 3 two-hour sessions. Prior to the exercise, the students attended several sessions on how to use GeoGebra software, and were asked to solve problems involving geometric constructions.

In the first two sessions, the students were required to solve the problems individually in accordance with a proposed problem-solving procedure that included the steps involved, an explanation of the difficulties that might arise, and a comparison of paper and pencil and computer approaches to solving the problems. Students were also asked to describe and record their beliefs, feelings and mental blocks when solving problems.

The third session was devoted to common approaches and the difficulties arising when trying to solve the problems. A preliminary analysis of the results from the preceding sessions was available during this session.

The problem-solving results required a more thorough study of the subjects' cognitive and instrumental understanding of geometric loci. This was achieved with semi-structured interviews conducted with nine group volunteers. The interviews were divided in two parts: task-based questions about the problems, asking the students to explain their methodologies and a series of questions designed to obtain information about visual and analytical reasoning, and visualization and instrumental difficulties.

4 *A priori* analysis of experimentation/typology of geometric loci

We must consider the functional approach used in the DGS (for the analysis we have used 3.7 y 4.0 versions). Hence, we can distinguish three cases or typologies of problem-solving:

Type 1:
Problems in which the locus is directly defined, so that the functional approach can be straightforward. For instance: the problem 1 (Table 1). This problem describes the function that takes the point C in the circumference to its image by expressed construction: the function maps each point C to the centroid of the triangle ABC.

Type 2:
Problems that, in order to be treated with GeoGebra, must be translated into a functional model, although they are not originally expressed in these terms. This functional approach is necessary to use a DGS to solve the problem. In this case, if you want a student to solve the problem, you must give clues for instrumental deconstruction and non-iconic visualization. Problems of this type are problems 2, 4, 5 and 6. In this type of problems, we usually need to use auxiliary objects, sometimes for mathematical reasons, and sometimes for technical reasons (due to the characteristics of GeoGebra).

Next we detail aspects of instrumental deconstruction and iconic/non-iconic visualization related to problem 4 and 5.

Problem 4 (Table 1): Instrumental deconstruction: If we want a dynamical approach in this problem, we must represent the ladder properly. To draw a ladder that really slips along the wall, it is not enough to draw a simple segment. We must use an auxiliary circle, determining the fixed length of the ladder. With this proper representation of the ladder, we can use the "locus" tool in GeoGebra.

Iconic/non-iconic visualization: The representation of this problem is iconic, it is easy to imagine and draw a ladder slipping along the wall. But the point is that to construct a good representation of the ladder, we need to use auxiliary (non-iconic) objects, as a circle, because the direct approach (considering the ladder as a general segment) does not allow us to produce a dynamic representation. (See comments below, on students' results). In this case, the auxiliary objects are needed for technical reasons. In GeoGebra, a segment is given by two points and if we move one of the points (say, along the wall) the segment just extends or shrinks. To obtain a segment that behaves as a ladder, we need a "technical" auxiliary construction. On the other hand, we must be precise in the definition of the midpoint, or any point of the ladder, this time for mathematical reasons. Hence, students must take care in the modelization of the problem, for technical and mathematical reasons.

Problem 5 (Table 1): Instrumental deconstruction: In a locus problem, we always need a "mover". In this case, the "mover" is the distance, that is, we must consider an auxiliary point on a segment that defines the distance between A and B. This is the key point in the construction. In GeoGebra 4.x we can also use a slider.

Iconic/non-iconic visualization: It is easy to solve the problem in an analytic way, but it is difficult to represent it visually, even with pencil and paper. To obtain a good representation using a DGS, we need to use a non-iconic representation, considering the distance as a "mover" of the locus. Before attacking the problem, students must perform a discursive reasoning, that is, before we represent anything, we must set the modelization clearly. Once the model is clear, the construction is quite straightforward.

> *Type 3:*
> *There are classic problems in which we can use GeoGebra to illustrate, but we cannot use the Locus tool to solve them*. We are not allowed to use the Locus command because it needs a "tracer" point and "mover" point, and in these types of problems there is no "mover" point because the tracer is determined by certain algebraic conditions. For example: problem 3 (Table 1): The construction is simple and the question is quite natural. However, the answer is not simple or obvious at all. This is an example of "nonparametric" locus and not solvable by GeoGebra (for the moment). Conceptually, it is a different example of locus, which establishes a clear difference with the "parametric" examples. Nevertheless, the students can guess the solution moving the point P, and considering an auxiliary line (passing by Q1 and Q2) to check if the three points are aligned.

❺ Results: episodes of visualization

Due to space constraints, we only present results for problem 4. This is a medium-advanced level problem for the student. The statement does not give explicit instructions for the construction. The situation is realistic and easy to understand, but the GeoGebra construction is not evident. We need to use an auxiliary object, and a mathematization process is demanded. Although the mediation of GeoGebra can help to make conjectures, it does not reveal the rationality below the calculations.

The visual-analytic reasoning demands to overcome the initial difficulty building the ladder. Using an auxiliary object, Geogebra produces a precise representation of the locus. Locus tool does not produce an algebraic answer, to obtain it we must take 5 points on the locus and then the conic passing by the 5 points: we now obtain an algebraic equation. For instrumental reasoning there are two key moments in the problem: (1) The construction of the ladder with an auxiliary circle (Fig. 1), and (2) if we want to study the locus of the positions that a point in the ladder describes, the point must be defined in a precise way (middle point, "1/4 point"). In order to use the locus tool, we must to take care when we choose the point on the ladder. We cannot just use a "point in segment", we must use the "middle point" tool, or a more sophisticated construction.

As mentioned above, both moments are different: while the precision in moment (2) is due to mathematical reasons, moment (1) is necessary for technical reasons. The definition of the element "segment" in GeoGebra is not suitable for modelizing a ladder.

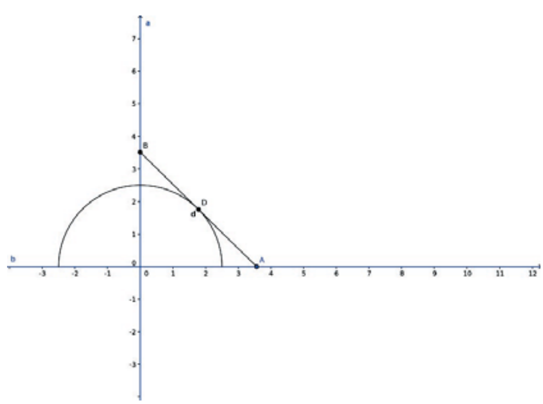

Figure 1. Solution Problem 4

5.1. *Typology of difficulties in the group*

A first type of difficulty is determined by static constructions (discrete treatment, Fig. 2). In this typology, students use GeoGebra as an advanced blackboard, but they do not use dynamic properties. They repeat the constructions for a number of points. To draw the geometric locus, they use the tool “conic by 5 points”.

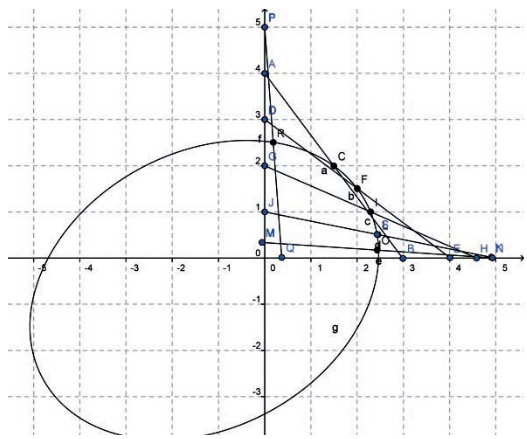

Figure 2. Student 13, solution problem 4

In these cases of iconic visualization, semiotic form leads to a misinterpretation of the dynamic figure by the discrete figural expansion. A non-iconic visualization and an appropriate instrumental deconstruction require some competences from the students: they are able to handle physical and mental representation, but the logic of the construction of a dynamic figure with a DGS is different.

A second type of difficulty is the incorrect definition of construction (use of free points). The students solve the problem (in an imprecise way), but this kind of solution implies that the tools in GeoGebra cannot be used. To use the “locus” tool, it is necessary that the defining points are correctly determined (they cannot be free points). With this approach, in the best case, the students can obtain a partially valid construction, but, as we can’t use the GeoGebra tools, we can’t obtain an algebraic answer. In problem 4, the difficulty lies in defining the point of the ladder that is not the midpoint. We obtain an approximate visual solution (using the “trace” tool) which is not usable with GeoGebra to produce a real locus (Fig 3). The students in this typology are absolutely convinced that their solution is right. They are not at all aware that a problem exists with the solution.

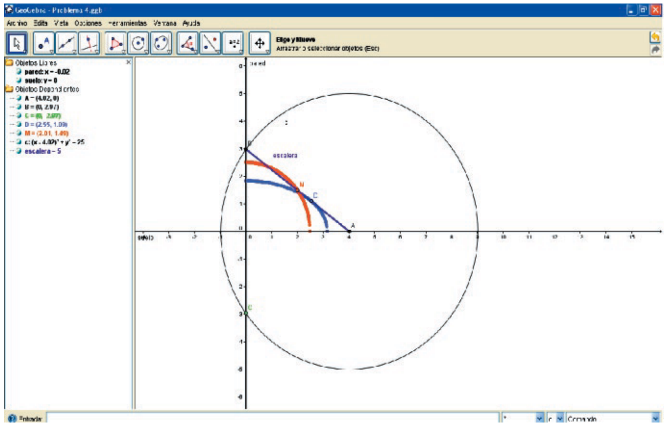

Figure 3. Student 23, solution problem 4

Finally another type of problems is the use of invalid instrumental elements. For example, some students use the “slider” tool to displace the “mover point”. The student realizes that the “mover point” must be controlled, and the control is done by the slider. The problem is that, for GeoGebra (before 4.x version), the slider is a scalar so it can’t be used with the locus tool. Again, students use the “trace” tool.[4]

5.2. *Case study*

To deepen the relationships that occur between instrumental genesis and figural genesis, we conducted a case study. Here, we illustrate the situation with two students who do not use the “locus command” for solving problem 4. This phenomenon occurs due to a difficulty in the articulation between the instrumental and figural genesis. The criteria to choose these students are based on their mathematical performance level, their visual cognitive style or preference for visual thinking, beliefs and feelings toward computer learning, and beliefs and feelings about visual thinking (Table 2). In the presented cases, we observe two different strategies of instrumental deconstruction, based on the different starting points of the subject.

It is necessary to obtain equilibrium between the dimensional deconstruction (expressed in the algebraic-analytic part, that allows us to perform the representation) and the instrumental deconstruction (expressed in the non-iconic visualization that allows the visual control in a DGS).

[4] *http://www.geogebra.org/help/docues/topics/746.html* From version 4.0, you can use the slider for a locus construction.

In Alberto's case, a reproduction of a physical form exists, from a visual/perceptive control and an implicit theoretical control. A control of the specific auxiliary objects exists, in both mathematical and technical-instrumental senses. In the second prospective teacher, Ana, the representation of the geometric object is based on an a priori theoretical control, and in GIII, although with a smaller iconic visual control. The control is a priori theoretical, and there is an axiomatic control, but there is no visual control. There is no real control of the dynamic phenomenon.

TABLE II
Criteria for case study

Case	*Gender*	*Mathematical achievement*	*Visual style*	*Beliefs about computer learning*	*Feelings about computers*	*Beliefs about visual thinking*	*Feelings about visualization*
Alberto	Male	High	Visualizing student	Positive	Likes	Positive	Likes
Ana	Female	High	Non-visualizing student	Positive	Dislikes	Positive	Dislikes

5.2.1. Alberto's Case

Alberto is a visualizing student. His pleasure or liking for mathematical visualization is closely intertwined with the evolutionary conception of mathematics: "The fun and intuitive character of mathematics is developed by a visual reasoning rather than through an algebraic reasoning, even though it is ideal that they are complemented in the problem solving process." (Alberto's questionnaire).

He considers that visual reasoning is essential in problem solving. The feeling of pleasure that is being experienced using visualization corresponds to the experience of control and generation of in-depth learning that is being experienced. He considers that it helps in its intuitive dimension of knowledge and in the formation of mental images.

Next, we illustrate the analysis of Problem 4 that we conduct about the subject's processes of visualization, representation and use (Annex 1).

The student first looks for the mathematical object, but he becomes blocked (Annex 1, 1). He even attempts a physical construction. This allows him an iconic visualization that turns into a visual and semiotic exploitation within the environment of dynamic geometry. For instance, in Annex (1) when the student ”search for mental image”, we see a process of “implemented discovering” in which the student seeks empirically, as if his figures were objects of experimentation, anticipating satisfactory loci. In this case we can recognize moments of completing the inductive reasoning by analytical reasoning indications. Other times, this analytical reasoning is explicit like in Annex 1 (10).

The student uses the visual power of the technology to have a better understanding of the situation mathematically and to have a context change to facilitate notion and property applications. GeoGebra works as a real tool in mathematical modeling. He is a student with a proficient knowledge of image use: concrete, kinesthetic and analogical illustrations.

5.2.2. *Ana's case*

The behavior of this student shows us some limits about visual apprehension. Although this student indicates that visual register was used for problem solving, she highlights that she does not like to solve the problem with a computer: “I'm not very excited about it, because I do not use it frequently. I think it would be sufficient to know the language (and I already know the analytical resolution) and to do as much an exercise per week to remember the language and to use it with students, in case you have time and/or you need it”. She gives more value to analytical thinking than visual thinking and she attributes less value to solve problems with computer.

There exists an *a priori* theoretical and axiomatic-analytical control, but there is no visual control. There is no control of the dynamic phenomenon. In this case, the dominant part is the non-iconic visualization, supported by analytical elements. However, this theoretical control does not add visual control on the DGS space (as it does on paper, see Annex 2, 7). She uses the GIII system, but she does not make that interpretation within the dynamic environment.

In summary, the activity of these students on visualization and instrumental deconstruction emphasizes individual differences coming from different cognitive styles and beliefs about computer use. Both students do not use the “Locus” command, but the data from their personal geometric workspace highlight essential differences in the interaction between figural and instrumental genesis. The discursive-analytic activity, for students whose geometric work relies on it, becomes an obstacle (Ana's case). However, the lack of it is also a problem, as discussed in a previous section. The control of images is a key property in visualization and for its use in the resolution of the problem. Among the features of DGS, visual apprehension must consider the theoretical objects as building tools, for which the instrumental genesis has a central role, as evidenced in Alberto's case.

6 Conclusion

From our study, some characteristic points of the Space for a Geometric Work (SGW) at teacher training, implemented with DGS, can be drawn. This paper focuses on geometric locus activities, identifying how figural and instrumental geneses are articulated in the personal SGW. Then, the emphasis is posed on the necessity of developing, for visualization in DGS, a balance between both non-iconic visualization and instrumental genesis, where instrumental deconstruction is crucial. The "appropriate" SGW appears particularly unstable and dependent on the students' visual cognitive style and beliefs about mathematic learning in computerized environments. In the observed examples, we show the necessary equilibrium between the dimensional deconstruction (expressed in analytical-algebraic analysis that allows the representation) and the instrumental deconstruction (expressed in non-iconic visualization that allows visual monitoring in a DGS).

In a large proportion of students, a heuristic deficiency exists in the geometric interpretation of visualizations that lead to the understanding of locus from the functional point of view. The data show that the heuristic function of visualization (or non-iconic visualization) requires visual deconstruction of the basic perceptive forms imposed in the first sight. However, there are a percentage of students (25%) whose tendency is the visual evidence, this being an obstacle (section 5.1). In the instrumental genesis, there is a procedimental dimension, in which we can see the necessary differentiation between technical and mathematical auxiliary objects (section 4). This aspect becomes a great difficulty to a great part of students, when they try to use the "Locus" tool, considering the functional approach. The blocking experiences and visualization difficulties in students appear in the production of interactive images and the use of analogical visualization.

Finally, the reorganization of the SGW seems more and more managed by a teacher accompanying the perceptual apprehension of students in order to get an operative apprehension (Duval 2005). We noticed that when we work on perceptual apprehension, using DGS, that is used to see figures and iconic visualization, we do not always reach to operative apprehension. That is, there are students with mathematical control, but that have great problems to produce dynamic figures. That is why we affirm the need for *progressive modeling in visualization*, and the need to introduce a progression in developing and in the manipulation of different types of representation, as a result of thinking through doing a modeling progress involves rational transitions along several dimensions (modes of signification from iconic to indexical to symbolic, ways to examine and to use representation, largely reflected in the case of Alberto). In Problem 4, we have detected limitations in visual apprehension of the resolutions of the students that, through a progressive modeling of visualization by the teacher, could be overcome. One proposal is to guide the construction performed by the students. One work in progress on this subject is shown in *http://www.mat.ucm.es/catedramdeguzman/drupal/igm/igm/materiales/pimcd239*. This is a new teaching module, connecting GeoGebra applets and web pages using javascript. This module could be proposed to be used for in-service mathematics training teachers.

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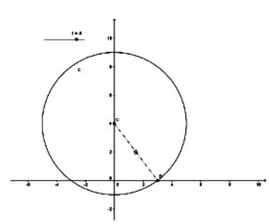
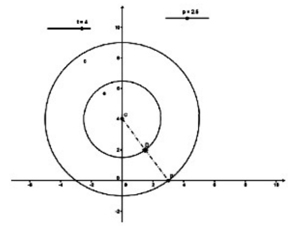
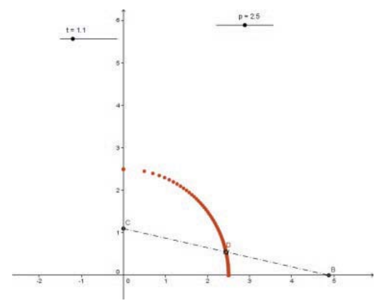
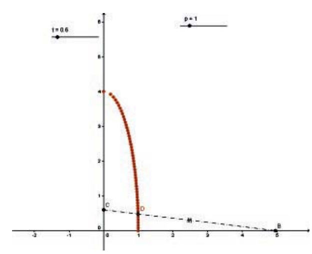
Annex 1

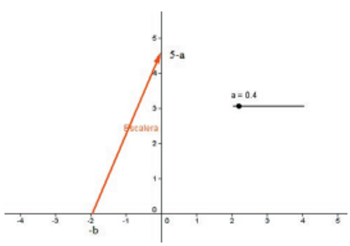
Analysis of the Alberto solution process, use of images in situation reported by the student in his protocol

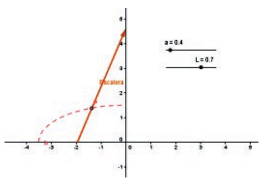
Method description, register visualization	*Typology of the use of representation/image/*
(1) In the first place, I make a representation of the problem on the paper. I try to look for a way to solve it analytically but I do not find any. I reflect on the possible relationships between the triangles that the ladder gradually generates as it slides downward against the wall and the floor without clearly reaching anywhere.	Drawing (of patterns and lines/figure) Analytical (Search for mental image (specific figure/illustration and dynamic image)
(2) I think about the answer: Will it be a straight line, an ellipse or a circle?	
(3) I left the problem for another day. I was thinking about it while I was doing other activities. I trusted my subconscious to continue to job.	Search for mental image
(4) I retake the problem with excitement and hope. I experiment with a pen and an elastic rubber rolled on its middle part. It seems to form an arc of a circle. At least, I already have an idea.	Physical manipulation - kinetics (Kinesthetic manipulation) Mental image – identification mathematical object
(5) I start to work with GeoGebra. After trying some construction with straight lines, I notice that the ladder being a segment of length 5 allows me to make a construction based on a circle of radius 5 that runs the y- axis.	Technological manipulation with the computer Representing radius of the circle (Specific illustrations)
(6) I generate a slider t and I define the center of the circle C = (0,t). The slider will shrink from 5 to 0. It is zero when the ladder lies on the ground. Point B represents the intersection of the circle and the x-axis.	 Interactive image generation, slider (Analogical)

(7) Once the ladder has been represented, I constructed another circle with variable radius (according to a new slider p). This circle indicates the point whose path will be our object of study. In our first case, it is the midpoint of the segment representing the ladder.	 Interactive image generation, slider (Analogical)
(8) I observe the path taken by the midpoint of the ladder upon activating the trace.	 Specific illustration with interactivity (Analogical)
(9) I am dealing with an arc of a circle with the center C = (0,0) and radius r = 2.5. Now, I try it with a point located 4 m from the starting endpoint of the ladder (positioned vertically over the y-axis at the start). I have obtained an arc of the ellipse with major vertical semi-axis of 4 and minor semi-axis of 1.	 Specific illustration with interactivity (Analogical)
(10) The geometric locus described by point D is an ellipse with major semi-axis: max (h, 5-h), Vertical if h > (5 – h), Horizontal if h < (5 –h)), and Minor semi-axis; Min (h, 5-h); horizontal if h> (5 – h), Vertical if h< (5 – h)), where h is the distance of the point from the base of the ladder that is in vertical position. In the case of h=2.5, both semi-axes are equal that confirms that it is a circle.	Analytical-visual Formulaic typology from the memory

Annex 2

Analysis of the Ana solution process, use of images in situation reported by the student in her protocol

Method description	*Typology of the use of representation/image*
(1) As the problem asks for the solution for the midpoint and for any point P of the ladder, I will first solve it for P and then to particularize to the midpoint.	
(2) I will assume that the ladder was in the beginning, just totally vertical and that it will finish completely horizontal (to see the largest possible path P).	mental image
(3) To write how the ladder go down I define the variable $a \in [0,5]$; by the Pythagoras Theorem $b^2 + (5-a)^2 = \text{Ladder}^2 = 5^2$, then $b = \sqrt{(10a - a^2)}$; in this way, the ends of ladder will be the points (-(b), 0) y (0, 5 - a)	She does not leave graphic register. mental image / non-iconic visualization
 (4) Now put a point P=(Px, Py) on the ladder: If $P_y = ((5-a)/b)P_x + 5 - a$, P will be on the line containing the ladder	Specific illustration with interactivity (analogical) Analytical-visual
(5) P is limited to the segment ladder, I consider L in [0,1] y $P_x = -Lb$ Thus, a point P on the ladder is written as: $P = [-Lb, (5-a)(1-L)] = [-L\sqrt{(10a - a^2)}, (5-a)(1-L)]$	Analytical reasoning
(6) Now I wonder: what is the way of P when I move a? The answer is given by I: $x = -L\sqrt{(10a - a^2)}$; II: $y = (5-a)(1-L)$	Analytical reasoning

(7) I try to write y depending on x. From (I) I obtain a = (1/L)*[5L ± √(25L²-x²)]; Substituting this value of a in (II), I obtain III: y = (5 - (5 L ± √(25 L² - x²)) / L) (1 - L); which describes an ellipse.	Analytical reasoning
(8) In this ellipse, I only consider the part corresponding to $x \leq 0$, $y \geq 0$, which is the part where I draw the ladder:  If I choose the midpoint, I take L=0.5, and we obtain the equation of a circumference.	Analytical Specific illustration with interactivity (analogical)