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PATTERN TASKS: THINKING PROCESSES USED BY 6TH GRADE STUDENTS

RESUMEN

Este documento es una descripción de un estudio en curso enfocado en tareas modelo de exploración y generalización, analizando el desempeño de cincuenta y cuatro alumnos de 6° grado cuando resuelven este tipo de tareas. El principal objetivo es entender las estrategias que utilizan, las dificultades que emergen del trabajo de los alumnos y averiguar el papel que desempeñan mediante la visualización en su razonamiento. Hasta ahora, los resultados indican que, en general, los alumnos tienden a usar planteamientos numéricos en lugar de planteamientos visuales. También tienden a usar estrategias incorrectas cuando intentan generalizar, siendo la más común un uso incorrecto de la proporción directa.

PALABRAS CLAVE:

- *Patrones*
- *Generalización*
- *Estrategias*
- *Visualización*

ABSTRACT

This paper gives a description of an ongoing study focused on pattern exploration and generalization tasks, analysing the performance of fifty-four 6th grade students when solving this type of tasks. The main goal is to understand the strategies they use, difficulties that emerge from students' work and ascertain the role played by visualization in their reasoning. Results so far indicate that, in general, students tend to use numeric instead of visual approaches. They also tend to use incorrect strategies when attempting to generalize, the most common being an incorrect use of direct proportion.

KEY WORDS:

- *Patterns*
- *Generalization*
- *Strategies*
- *Visualization*

RESUMO

Este trabalho apresenta uma descrição de um estudo em curso focado na exploração de padrões e tarefas de generalização, analisando o desempenho de 54 alunos do sexto ano ao resolver este tipo de tarefas. O objetivo principal é compreender as

PALAVRAS CHAVE:

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estratégias que eles usam, dificuldades que emergem do trabalho dos estudantes e verificar o papel desempenhado pela visualização no seu raciocínio. Os resultados obtidos até agora indicam que, em geral, os estudantes tendem a utilizar abordagens numéricas, em vez de abordagens visuais. Eles também tendem a usar estratégias incorretas ao tentar generalizar, sendo a mais comum o uso incorreto de proporção direta.

RÉSUMÉ

Cet article analyse les performances de 54 élèves de 6ème année (11-12 ans) lorsqu'ils doivent résoudre des problèmes impliquant un examen et une généralisation de modèles. L'objectif principal de cette étude toujours en cours est de comprendre les stratégies qu'ils mettent en œuvre, les difficultés auxquelles ils sont confrontés et de déterminer le rôle joué par la visualisation dans leur raisonnement. Les résultats, jusqu'à présent, montrent que les élèves ont tendance, en général, à privilégier les approches numériques plutôt que visuelles et à utiliser des stratégies erronées (mal utiliser la proportion directe est la plus commune d'entre elles) lorsqu'ils essayent de généraliser.

MOTS CLÉS:

- *Modèles*
- *Généralisation*
- *Stratégies*
- *Visualisation*

1. Introduction

A quarter of a century ago, problem solving became a focus of school mathematics, and continues to do so. According to recent curricular guidelines in several countries, one of the main purposes of mathematics learning is to develop the ability to solve problems. In spite of the curricular relevance given to this theme over the last few years, several international studies (SIAEP, TIMSS, PISA) have shown that Portuguese students perform badly when solving problems (Ramalho, 1994; Amaro, Cardoso & Reis, 1994; GAVE, 2004). These results, along with the difficulties observed in classroom experiences, constitute a matter of concern to the community of researchers and educators.

Pattern exploration tasks may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data in a systematic way, conjecturing and generalizing. Work with numeric, geometric and pictorial patterns may be helpful in building a more positive and meaningful image of mathematics and contribute to the development of several skills related to problem solving and algebraic thinking (NCTM, 2000; Vale, Palhares, Cabrita & Borralho, 2006). On the other hand, Geometry is considered a source of interesting problems that can help students develop abilities such as visualization, reasoning and argumentation. Visualization, in particular, is an essential mathematical capacity, especially where problem solving is concerned. According to Polya (1945), visual representations are often used as a strategy that allows powerful and creative solutions. Despite this implicit recommendation, and according to some studies, visual approaches are not very common in students' mathematical experiences (Healy & Hoyles, 1996; Presmeg, 2006). Although the usefulness of visualization and graphical representations is being recognized by many mathematics educators, in Portuguese classrooms, teachers tend to privilege numeric aspects over geometric ones (Vale & Pimentel, 2005). Taking all of this into consideration, we think that more research is still necessary concerning the role images play in the understanding of mathematical concepts and in problem solving. It is also important to ascertain when visualization is more useful than analytical methods (Gutiérrez, 1996).

This study intends to analyse difficulties and strategies that emerge from the work of 6th grade students (11-12 years old) when solving problems involving pattern seeking and the role played by visualization on their reasoning. The tasks used in the study require pattern generalization; however students of this age have not yet received formal algebra instruction, thus the importance of analysing their approaches. This study attempts to address the following research questions:

1) Which difficulties do 6th grade students present when solving pattern exploration tasks?

2) How can we characterize students' strategies?

3) What is the role played by visualization on students' reasoning?

2. Theoretical framework

2.1. *Patterns in the teaching and learning of mathematics*

Many mathematicians share an enthusiastic view about the role of patterns in mathematics, some even consider mathematics as being the science of patterns (Devlin, 2002; Steen, 1990) which highlights the transversal nature of this theme.

The search for patterns is seen by some as a way of approaching Algebra since it is a fundamental step for establishing generalization, which is the essence of mathematics (Mason, Johnston-Wilder & Graham, 2005; Orton & Orton, 1999; Schoenfeld & Arcavi, 1999; Zazkis & Liljedahl, 2002).

Searching for patterns in different contexts using and understanding symbols and variables that represent patterns and generalizing, are significant components of the mathematics curriculum in many countries. The Portuguese curriculum mentions the importance of developing abilities such as searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing, and generalizing (DEB, 2001; ME-DGIDC, 2007). These abilities are directly related to algebraic thinking but also support the development of mathematical reasoning, communication and connections between mathematical ideas (NCTM, 2000).

2.2. *The nature of mathematical thinking*

Patterning activities can be developed in a variety of contexts (numeric, geometric, pictorial) and through the application of different approaches. Gardner (1993) claims that some individuals recognize regularities spatially or visually, while others detect them logically and analytically. In fact, it is common, in mathematical activities, that different individuals process information in different ways. Many students favour analytical methods, while others have a tendency to reason visually.

A study developed by Krutetskii (1976) with a sample of mathematically gifted students showed that they used a variety of different approaches in problem solving. While analysing the type of reasoning used by those students, Krutetskii (1976) identified three main categories: analytic (non- visual), geometric (visual) and harmonic (use of the two previous types of reasoning).

In spite of the existence of different approaches to the same problem, most students prefer to use numerical relations as a support for reasoning, perhaps reflecting the work promoted in the classroom where analytic representations prevail. However some studies indicate that most students are more successful when they use a harmonic or mixed approach (Moses, 1982; Noss, Healy & Hoyles, 1997; Stacey, 1989; Becker & Rivera, 2005). The relationship between the use of visual abilities and students' mathematical performance constitutes an interesting area for research. Many researchers stress the importance of the role visualization plays in problem solving (Presmeg, 2006;

Shama & Dreyfus, 1994), while others claim that visualization should only be used as a complement to analytical reasoning (Goldenberg, 1996; Tall, 1991). In spite of some controversy, these visions reflect the importance of using and developing visual abilities, enhancing students' mathematical experiences.

According to Presmeg (2006), teachers tend to promote visual reasoning as a possible strategy for problem solving only at an initial stage. Several studies point to the potential of visual approaches for supporting problem solving and mathematical learning. The reality of our classrooms, however, tells us that students frequently display some reluctance to exploit visual support systems (Dreyfus, 1991) and tend not to make links between visual and analytical thought (Presmeg, 1986). These ideas imply that the role of visualization in school mathematics should be re-evaluated and there are various reasons pointing to the importance of visualization: (1) mathematics is currently identified with the study of patterns which, together with the use of technology, may diminish the difficulty of algebraic thinking; (2) visualization can often provide simple and powerful approaches to problem solving; (3) teachers should recognize the importance of helping students to develop a repertoire of different techniques to approach mathematical situations (Thornton, 2001).

2.3. *Students' thinking processes in pattern generalization*

There has been significant research in relation to students' difficulties and generalization strategies, from pre-kindergarten to secondary school, when solving problems requiring pattern exploration. The results that emerged from some of these studies are discussed in this section. Despite the focus of our study being on 6th grade students, we think it is worthwhile to analyse conclusions and perspectives from different authors, approaching different levels.

Stacey (1989) focused her research on the generalization of linear patterns by students aged 9-13 years old. She classified students' strategies when solving contextualized linear generalization tasks, whether or not they obtained correct answers. Strategies found were: *counting*, *whole-object*, *difference* and *linear*. In the *counting* strategy, students counted the number of items in a figure. Those who employed the *whole-object* strategy used a multiple of a previous value, assuming the problem implied direct proportion. The *difference* strategy consisted of using a multiple of the difference between two consecutive items of the sequence. Finally, students who used a *linear* strategy applied a linear model to find solutions. In her study, Stacey (1989) concluded that a significant number of students used an incorrect proportional method when attempting to generalize. She also reported some inconsistencies in the strategies used by students in *near generalization* tasks (questions that can be solved by the use of a drawing or of a recursive method, for example finding the 5th item of the sequence) and *far generalization* tasks (strategies stated before are no longer adequate, these questions imply the finding of an explicit rule). She concluded that visual representations had a major influence on their approaches, although she didn't explore it further.

García Cruz & Martinón (1997) developed a study that aimed to analyse the processes of generalization developed by secondary school students. Their categorization of the methods used by these students was based on Stacey's work. They considered three main categories: *counting* (including counting the items on a drawing and extending a sequence using a recursive method), *direct proportion* and *linear*. They also classified strategies according to their nature: *visual*, *numeric* and *mixed*. If the drawing played an essential role in finding the pattern it was considered a *visual* strategy; on the other hand, if the basis for finding the pattern was the numeric sequence then the strategy was considered *numeric*. In *mixed* strategies the students acted mainly on the numeric sequence and used the drawing as a means to verify the validity of the solution. Results of this research have shown that drawing played a double role in the process of abstracting and generalizing. It represented the setting for students who used visual strategies in order to achieve generalization and, on the other hand, acted as a means to check the validity of the reasoning for students who favoured numeric strategies.

Orton & Orton (1999) focused their research on linear and quadratic patterns with 10-13 year old students. They reported a tendency to use differences between consecutive elements as a strategy in the generalization of linear patterns, and its extension to quadratic patterns, by taking second differences, but without success in some cases. They also indicated as obstacles to successful generalization, students' arithmetical incompetence and their fixation on a recursive approach which, although being useful in solving near generalization tasks, does not contribute to the understanding of the structure of a pattern.

Sasman, Olivier & Linchevski (1999) developed a study with 8th grade students, working with generalization tasks that involved different representations. Results showed that students almost exclusively used number contexts, neglecting drawings, and favoured a recursive method, making several mistakes related to an incorrect use of direct proportion.

In a more recent study, Becker and Rivera (2005) described 9th grade students work after they performed generalizations on a task involving linear patterns. They tried to analyse successful strategies students used to develop an explicit generalization, and to understand their use of visual and numerical cues. The researchers found that students' strategies appeared to be predominantly numeric and identified three types of generalization: *numerical*, *figural* and *pragmatic*. Students using numerical generalization employed trial and error with little sense of what the coefficients in the linear pattern represented. Those who used *figural* generalization focused on relations between numbers in the sequence and were capable of seeing variables within the context of a functional relationship. Students who used *pragmatic* generalization employed both numerical and figural strategies, seeing sequences of numbers as consisting of both properties and relationships.

Analysing the results of these studies we may conclude that, in spite of being developed in different contexts, they present a series of common factors, concerning both: the nature of the strategies emerging from students' work, proposing similar categories and concluding the preference for the application of numeric strategies; and the types of difficulties presented, as the erroneous use of direct proportion and the tendency to think recursively that makes it harder to generalize. Since most of the categorizations discussed in this section were based on Stacey's work, which presents a set of well defined and refined generalization strategies drawn from students aged 9-13 years old, close to the ages of the students participating in this study, we chose to use this framework when analysing students' answers.

3. Method

In this study we chose a mixed methodology (Creswell, 2003), predominantly qualitative and interpretative. Qualitative data was obtained through clinical interviews, participant observation and from documents produced by the students involved. The main purpose was to gain some insight on their thinking processes and difficulties. To complement and clarify these results we collected quantitative data from tests.

Fifty-four sixth-grade students (11-12 years old), from three different schools in the North of Portugal, participated in this study over the course of a school year. The study was divided into three stages: the first corresponded to the administration of a test focusing on pattern exploration and generalization problems; the second stage, which lasted nearly 6 months, involved all students in each classroom working in pairs, solving patterning tasks focussed on near and far generalization; and, during the third stage, students repeated the test in order to examine the impact of the work carried out during the second stage on their ability to generalize. These students had no prior experience with these kinds of tasks and were described by their teachers as being of average ability. In order to organize students in heterogeneous pairs, we used the results from the first application of the test, along with the teachers' views of their students' mathematical skills. Over the school year all students involved in the study solved 7 tasks and two pairs from each school were selected for clinical interviews. As we wanted to gain insight and acquire an in-depth perspective regarding complex thinking processes, we considered the case study approach to be the best option (Yin, 1989). These pairs of students were chosen based on their oral communication skills, their willingness to be interviewed and the variety of strategies that they applied in the pre-test. Student activity when solving the tasks was videotaped and transcribed for further analysis in order to investigate their mathematical reasoning, in particular, strategies used to solve each of the problems posed, as well as difficulties they experienced on that particular activity.

4. Results

In this paper we will focus on the results of the pre-test, categorizing data in relation to students' generalization strategies. The fact that this is an ongoing study supports this option, in addition to hte fact that the pre-test allowed us to retrieve significant data during a stage where students had no experience with pattern exploration.

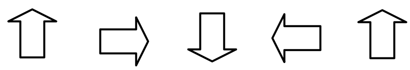
At the beginning of the study students were given a written test with pre-algebraic questions. The test contained sixteen introductory questions consisting of visual and numerical sequences, including cases of repetition and growth (see *a*) for an example), followed by two tasks involving *near* and *far generalization* (see *b*) and *c*)). With the first set of questions we intended to analyse the ability to interpret and continue different types of sequences, recognizing the subsequent elements. In the second and third tasks, students had to solve problems that implied finding a pattern, to discover terms in near and far positions, explaining, in these cases, their thought process.

a) Examples of introductory questions:

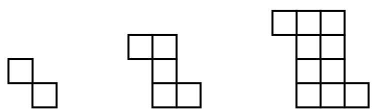
1. *Complete the following sequences indicating the next* two *elements:*

1.2: 2, 5, 8, 11, 14

1.7:

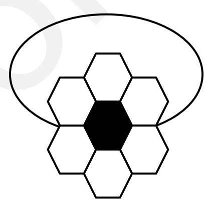


1.13:



b) Second task:

2. *Joana likes to make necklaces using flowers. She uses white beads for the petals and black beads for the centre of each flower. The figure below shows a necklace with one flower and a necklace with two flowers, both made by her.*



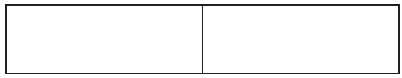
2.1. How many white and black beads will Joana need to make a necklace with 3 flowers? Explain your conclusion.

2.2. How many white and black beads will Joana need to make a necklace with 8 flowers? Explain your conclusion.

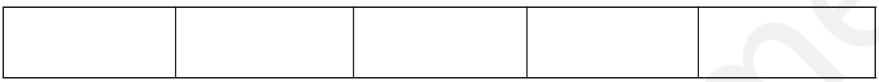
2.3. If Joana wants to make a necklace with 25 flowers, how many white and black beads will she need? Explain your conclusion.

c) Third task:

3. *On the following figure you can count three rectangles.*



Consider the figure below:



3.1. *How many rectangles of different sizes can you find?* Explain your reasoning.

3.2. *If you had 10 rectangles in a row, how many rectangles of different sizes could you count?* Explain your reasoning.

We decided to start the test with some simple patterning tasks, including extending different types of sequences, approaching a variety of contexts: numeric and visual patterns, repetition and growth patterns, increasing and decreasing patterns, linear and quadratic patterns. We did not ask for an explicit rule, we just asked students to continue the given sequences, indicating the next two elements. These kinds of tasks involved near generalization and allowed us to analyse how these students understand patterns in different contexts, as well as their level of success.

The problems presented in the second and third tasks of the test required students to engage in near and far generalization. The bead problem represents an increasing linear pattern, presented in a visual context. There were two main reasons for the inclusion of this particular task in the test: it allows the application of a diversity of generalization strategies, numeric, visual or mixed (García Cruz & Martinón, 1997); and the observation of the structure of the figure was presumably enough to easily determine an explicit rule. The rectangles problem was more complex than the previous tasks. Despite being presented visually, the analysis of the figure alone wasn't enough to determine an explicit rule since it required organized reasoning and possibly the combination of visual and numeric approaches.

The test was developed to ascertain students' abilities when performing pattern seeking and generalization tasks and to characterize the strategies used to solve these types of problems. It was validated by a panel of teachers and researchers in mathematics education and was also solved by 5th and 6th grade students from different schools, prior to its implementation in this study. We also developed a holistic scale to evaluate the answers, establishing five levels of performance for each question, varying from 0 to 4. These levels were adapted to each task, considering that 0 is absence of answer or a totally incorrect answer and 4 corresponded to a correct answer with a clear explanation. The reliability of this instrument was measured using Cronbach's alpha, obtaining a coefficient of 0.845.

4.1. *Thinking strategies that emerged from the application of the test*

In the first task of the test, students only had to continue visual and numeric, repetition and growth sequences, indicating the next two elements. No explanation was requested since the goal was mainly to discover a rule to extend them. Therefore, the type of answers obtained in this particular task did not favour a detailed analysis of thinking strategies so, in this section, we focus on the second and third tasks of the test.

Considering the nature of the tasks and the analysis of the work carried out by the students, we felt the need to adjust Stacey's (1989) strategy categorization in order to describe, as accurately as possible, their reasoning. We established four main categories: *counting*, *whole-object*, *recursive* and *linear*. In some cases, we considered that a particular category had to be divided into different approaches due to the structure of reasoning presented.

Most of the tasks we designed for this study had a strong visual component, as we can see, for example, in the *beads* and *rectangles* tasks in the test. Near generalization questions can easily be solved by making a drawing of the requested term of the sequence and counting its elements, using what Stacey (1989) called the *counting* strategy (*C*).

The *whole-object* strategy (Stacey, 1989) also emerged from the work of some students. As we discussed earlier, this strategy is associated to direct proportion situations, once it considers multiples of a specific term of a sequence, and the problems presented in the test do not fit that model. For the strategy to be adequate, students had to make a final adjustment based on the context of the problem. In this respect, we identified two different ways in which students applied strategies associated to the *whole-object* strategy: using multiples of a given term of the sequence without any adjustment to the result (*W1*); using multiples of different terms of the sequence adding them at the end (*W2*) (in this case the requested term is obtained by decomposition, using known elements of the sequence).

These types of tasks can also promote the use of *recursive* thinking, especially when near generalization is involved. Therefore, it came as no surprise that some students used the common difference between two consecutive terms of the sequence to solve some of the questions posed. We distinguish two situations in which this strategy was employed: extending the sequence using the common difference (*R1*); using multiples of the common difference (*R2*). The application of this last strategy (*R2*), as it was described, without a final adjustment, led to incorrect answers.

The *linear* strategy (Stacey, 1989) relates to the use of expressions of the type $an+b$ ($b\neq 0$). In this study we considered four subcategories that are in some way linked to this particular strategy: identifying an explicit rule that relates the order of a given term of the sequence with the number of elements of that term (*L1*); using multiples of a given term of the sequence and making a final adjustment based on the context of the problem (*L2*); using multiples of a given term of the sequence and making a final adjustment based only on numeric relations (*L3*); using multiples of the common difference making a final adjustment based on the context of the problem (*L4*).

Table I summarizes the number of answers obtained in each of the categories described above (*counting*, *whole-object*, *recursive* and *linear*). In some cases we were not able to categorize students' answers because they were left it in blank or because their reasoning was confusing. Those cases appear in the last column of the table, as not categorized answers (*NC*).

TABLE I
Summary of students' responses

	C	W1	W2	*W*	R1	R2	*R*	L1	L2	L3	L4	*L*	*NC*
2.1	*14*	20	5	*25*	4	-	*4*	-	1	-	-	*1*	*10*
2.2	*4*	23	1	*24*	3	3	*6*	-	2	-	-	*2*	*18*
2.3	*1*	21	1	*22*	-	2	*2*	-	2	1	1	*4*	*25*
3.1	*38*	-	-	-	-	-	-	-	-	-	-	-	*16*
3.2	*23*	2	-	*2*	-	-	-	9	-	-	-	*9*	*20*

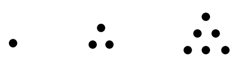
To better interpret Table I, some considerations need to be made. In order to make it easier to understand the structure of the pattern in the second task of the test, we presented an image of the first two elements of the sequence. In spite of this fact, only fourteen students used counting as a solving strategy, favouring instead a numerical approach such as direct proportion (W1). Four students made a drawing to solve the first two questions and applied direct counting to determine the number of beads, but they were not able to solve the last question using the same method, since it involved *far generalization*, so they left it in blank or a feeble attempt was made to solve it. The few students that successfully solved questions 2.1 and 2.2 of this task used *counting* or *recursive* reasoning (extending the given sequence, strategy R1). There was a general tendency to maintain the same strategy for the three questions of the second task. However some students started using the *counting* strategy, although for the far generalization, they shifted to a direct proportion model.

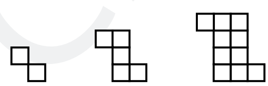
Students considered that the third task of the test was more complex than the others, possibly because they were not capable of translating the given context into numbers. This fact is consistent with the predominance of the *counting* strategy in both questions. The majority of students made a drawing of the situation and identified rectangles of different sizes, counting them. Some students identified the existence of different rectangles but, as they were unable to find an organized way to approach the question, they did not discover all the cases. In the second question of this task no figure was given. Most students started by representing the situation with a drawing, but in the end they were unable to discover the pattern due to the application of inadequate strategies: *counting* (using a confusing diagram), *whole-object* (considering a proportional model) or *linear* (considering that the rule to find the number of rectangles was $n+1$, n being the number of unitary rectangles).

4.2. *Difficulties emerging from the application of the test*

All the answers were categorized in an attempt to find and explain difficulties students experienced when solving the test.

Best results were achieved on the first task of the test, possibly because students had some prior experience solving these kinds of patterning activities. Nevertheless, some difficulties were found that should be pointed out. The task related to continuing repetition and growth patterns presented in different contexts. The structure of growth patterns implies that each element of the sequence is related to the preceding one, therefore these kinds of patterns lead to generalizations and their representation through variables. Results show that the students involved in the study were more successful with continuing repetition patterns than growth patterns. This may perhaps indicate that they had more previous experience with the first kind of sequences or perhaps, more likely, that the second type has a more complex structure. Although we did not expected it, some of the growth sequences were interpreted by several students as repetition patterns, both in visual and numerical contexts. The two most paradigmatic cases were the numerical sequence *1, 4, 9, 16* and the visual sequence . Students continued the first by adding 3 to 16 and 5 to 19, instead of obtaining the squares of whole numbers. In the second case, we expected to get a hexagon and a heptagon and some students presented a triangle and a square, repeating the given sequence. The majority of students achieved better results on the questions involving numerical patterns than on those involving visual patterns. They presented very low scores when completing the following two sequences, whose nature was mainly visual:





It is possible that the low scores obtained on the second sequence were due to simultaneous variation of the length and height and on the first sequence mainly because of the triangular arrangement of the dots.

On the second task students tended to inadequately use a direct proportion model, in some way familiar to them. This may indicate that they did not properly analyse the structure of the sequence, thinking of each flower as a disjoint unit. Most of them considered that each flower had six white beads and one black, so a necklace with eight flowers would have forty-eight white and eight black beads and a necklace with twenty five flowers would have hundred and fifty white and twenty five black beads. These students used the *whole-object* strategy (W1) and did not notice that consecutive flowers had two white beads in common. Perhaps they would easily see the mistake if they checked the rule with a drawing, verifying their answer. On the last two questions of this task, which involved far generalization, students' scores were very low. We think that students' tendency to manipulate only numbers contributed to increasing the difficulty of finding the pattern, which was noticed as the order of the term got higher.

Table II summarizes the results described above, relating levels of performance with the strategies used to solve each of the questions of the bead task, also indicating the number of students in each category.

TABLE II
Levels of performance obtained in task 2 of the pre-test

	Level 0		*Level 1*	*Level 2*		*Level 3*		*Level 4*	
2.1	Counting based on a wrong drawing	5		Recursive (R1)	1	Linear (L2)	1	Counting	9
	Whole-object (W1)	20							
	Whole-object (W2)	5						Recursive (R1)	3
	The reasoning isn't clear	8							
	Presents an answer without further explanation	2							
2.2	Counting based on a wrong drawing	3		Recursive (R2)	3	Recursive (R1)	1	Counting	1
	Whole-object (W1)	23							
	Whole-object (W2)	1							
	The reasoning isn't clear	11		Recursive (R1)	1	Linear (L2)	2	Recursive (R1)	1
	Presents an answer without further explanation	4							
	No response	3							

2.3	Counting based on a wrong drawing	1	Linear (L2)	2	Recursive (R2)	2		
	Whole-object (W1)	21	Linear (L4)	1				
	Whole-object (W2)	1						
	Linear (L3)	1						
	The reasoning isn't clear	10						
	Presents an answer without further explanation	3						
	No response	12						

The last task of the test turned out to be the most difficult for these students. Only one of the fifty four students in the study gave a correct answer to the first question. He used a diagram and the *counting* strategy in an organized way, but as we introduced a higher number of rectangles, in the second question, this strategy was no longer adequate. The majority of the students identified only the smaller rectangles and the bigger one, possibly influenced by the example given. In some cases, they used direct proportion to determine the number of rectangles, in a similar way to the previous task, considering that if they had ten small rectangles in a row, then they would have to duplicate the result obtained in the first question, where there were five. The use of proportional reasoning in these cases shows that numbers are manipulated without meaning, having considered that this type of model is appropriate to all situations.

Table III summarizes the levels of performance obtained in the third task, as well as the strategies used by students to solve each of the questions.

TABLE III
LEVELS OF PERFORMANCE OBTAINED IN TASK 3

	Level 0		*Level 1*		*Level 2*		*Level 3*	*Level 4*	
3.1	Presents an answer without further explanation.	4	Counting	28	Counting	9		Counting	1
	The reasoning isn't clear	6							
	No response	6							
3.2	Counting based on a wrong drawing	2	Counting	15	Counting	6			
	Whole-object (W1)	2	Linear (L1)	9					
	Presents an answer without further explanation	4							
	The reasoning isn't clear	6							
	No response	10							

Globally we consider that the scores were very low. The few students that obtained higher scores applied counting or recursive reasoning. They were not able to find a correct rule to solve far generalization questions, which explains the scarcity of linear strategies and their inadequate use.

It is obvious that these students experienced serious difficulties with two of the questions in this test, questions 2.3 and 3.2, either by not solving them or by using inadequate strategies. They had no prior ongoing experiences with generalization tasks, only with patterning activities that involved recursive reasoning like continuing a sequence, and they were used to privilege numeric approaches. The two mentioned questions required finding a rule, starting with a visual context, which may explain the outcomes. These results indicate that it is fundamental that students are motivated to think about these types of relations between variables, establishing generalization, and giving meaning to both numerical and visual contexts, by relating them.

4.3. *The role of visualization in students' reasoning*

From the results of the test, we can identify some tendencies concerning the role of visualization in students' reasoning. Since we only had access to their written work, no interviews were carried out at this stage of the study.

Students presented more difficulties continuing visual sequences than numeric ones. This could mean that their numeric abilities overcame spatial ones, perhaps due to their greater experience working with numeric contexts, as was outlined by their teachers.

It is also important to try to understand the implications of using visual strategies when solving problems involving pattern seeking. According to Presmeg (1986), a strategy is considered visual if the image/drawing plays a central role in obtaining the answer, either directly or as a starting point for finding the rule. In this sense, we believe that *counting* and *linear* strategies *L1*, *L2* and *L4* are included in this group. Direct *counting* over a drawing proved to be a useful strategy to solve near generalization questions in the beads problem, for those who made a correct representation of the sequence. However, students who employed this strategy correctly did not use those representations to find the structure of the pattern and generalize, presenting difficulties when far generalization was involved. The most frequently used strategy in the third task was also *counting*. As in the previous task, students were not able to find the pattern and generalize because the drawing was only made with the purpose of counting all the different rectangles without any implicit order. No student applied *L1* to solve the second task of the test. This strategy appeared only on the rectangles problem but the rule those students found was incorrect. We also consider linear strategies *L2* and *L4* as being visual once the adjustment is made based on the context. *L2* was only used by two students who first considered that a necklace with 8 flowers had white beads and then subtracted 14 common beads. Only one student used strategy *L4* to solve question 2.3, which required far generalization, but he was not able to make the right adjustment. We think that it involved a higher level of abstraction in visualization, which was difficult to attain at this stage of the study.

5. Discussion

In this research, the main purpose of using pattern exploration tasks was to set up an environment in order to analyse difficulties presented by students, strategies emerging from their work, and the impact of visual contexts in generalization.

As for the research questions outlined earlier in this paper, we can make the following observations: (a) a variety of strategies were identified in the work carried out by students, although some were more frequent than others, such as *counting* and *whole-object*; (b) students chose numeric approaches over visual ones, showing that it was a more familiar working context. However, visualization proved to be useful in situations such as doing a drawing and counting its elements; (c) students experienced several difficulties when solving problems involving pattern exploration, especially when they had to generalize for distant values. They achieved better results in near generalization questions than in far generalization questions; (d) students were not able to find adequate explicit rules, revealing difficulties in finding a functional relationship and making many mistakes like the application of a direct proportion model when not adequate.

At this point, this research contributes to the idea that students prefer analytic/numeric approaches to mathematical activities, converting into numbers even problems of a visual nature. However, as discussed earlier, studies relating to visualization and the role of mental images in mathematical reasoning have shown the importance of representations in conceptual development (Palarea & Socas, 1998). Therefore, it is important to provide tasks that encourage students to use and understand the potential of visual strategies and to relate number contexts with visual contexts in order to then understand the meaning of numbers and variables. Communication is also an essential part of mathematics education, which is considered as a transversal ability to be developed. In this respect, and as a general comment, it is important to reflect that only a few students presented their reasoning in a clear way and the majority did not even justify their answers, which gave us reason to think that they experienced difficulties in expressing their reasoning. Even though it was not a goal in this study, these findings suggest that we should pay more attention to communication in our classrooms, creating environments and opportunities to develop abilities related to this skill.

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