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A SEMIOTIC VIEW OF MATHEMATICAL ACTIVITY WITH A COMPUTER ALGEBRA SYSTEM

RESUMEN. En este artículo defendemos la tesis de que un marco semiótico permite entender mejor como el uso de un Sistema Computarizado para Algebra (CAS) puede ayudar, o limitar, la actividad matemática. Este trabajo se sitúa en un marco teórico en el que hacer y aprender matemática es considerado un comportamiento semiótico. Partiendo de la noción de signo triádico (representamen, objeto, interpretado) desarrollada por Peirce, afirmamos que el uso de los CAS para cambiar de representamen (representación) en el estudio de un objeto matemático puede ayudar al estudiante a producir varios interpretados (interpretaciones) para este objeto. Esos diferentes interpretados, basados en diferentes representaciones, permiten un acceso epistemológico al objeto. Utilizamos la distinción de Duval entre conversión y tratamiento para distinguir las diferentes formas de actividad semiótica con los CAS. Ilustramos este argumento mediante un extracto del diálogo entre dos estudiantes universitarios mientras resuelven un problema matemático usando las CAS.

PALABRAS CLAVE: Semiótica, sistema computarizado para álgebra, conversiones e tratamientos, signo de Peirce, polinomio de Mac Laurin.

ABSTRACT. I argue that a semiotic framework enables a rich understanding of how the use of a computer algebra system (CAS) may enable, or constrain, mathematical activity. This argument is rooted in a framework in which the doing and learning of mathematics is regarded as a semiotic endeavour. Using Peirce's notion of a triadic sign (representamen, object and interpretant), I argue that the ability of the student to move between different representamen (representations) of the same mathematical object with CAS may help the student generate different interpretants (ideas in the mind) for this object. These multiple interpretants, based on multiple representamen, enable epistemological access to the object. I use Duval's distinction between conversions and treatments to distinguish between the different forms of semiotic activity with the CAS. To illustrate my arguments I examine a vignette in which two undergraduate university students use CAS while solving a mathematical problem.

KEY WORDS: Semiotics, computer algebra system, conversions and treatments, Peirce sign, Maclaurin polynomial.

RESUMO. Neste artigo defendemos a tese de que um quadro semiótico permite perceber melhor como o uso de um Sistema de Álgebra Computacional (CAS) pode apoiar ou limitar a actividade matemática. Este trabalho situa-se num quadro teórico em que fazer e aprender matemática é considerado um comportamento semiótico. Partindo da noção de signo triádico (*representamen*, objecto, interpretado) desenvolvido por Peirce, afirmamos que o uso dos CAS para mudar de *representamen* (representação) no estudo de um objecto matemático pode ajudar o estudante a produzir vários interpretados (interpretações) para esse objecto. Esses diferentes interpretados,

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baseados em diferentes representações, permitem um acesso epistemológico ao objecto. Utilizamos a distinção de Duval entre conversão e tratamento para distinguir as diferentes formas de actividade semiótica com os CAS. Ilustramos este argumento através de um extracto do diálogo entre dois estudantes universitários enquanto resolviam um problema matemático usando os CAS.

PALAVRAS CHAVE: Semiótica, sistema de álgebra computacional, conversões e tratamentos, signo de Peirce, polinómio de Mac-Laurin.

RÉSUMÉ. Dans cet article nous défendons la thèse qu'un cadre sémiotique permet de mieux comprendre comment l'usage d'un programme informatique pour l'algèbre (CAS) peut aider, ou contraindre, l'activité mathématique. Nous nous plaçons dans un cadre théorique dans lequel faire et apprendre les mathématiques est considéré comme un comportement sémiotique. À partir de la notion de signe triadique (representamen, objet, interprétant) développée par Peirce, nous affirmons que l'usage des CAS pour changer de representamen (représentation) pour étudier un objet mathématique peut aider l'apprenant à générer différents interprétants pour cet objet. Ces différents interprétants, basés sur différentes représentations, permettent un accès épistémologique à cet objet. Nous utilisons la distinction de Duval entre conversion et traitement pour distinguer les différentes formes d'activité sémiotique avec les CAS. Cet argument est illustré par le protocole d'un dialogue entre deux étudiants universitaires qui résolvent un problème mathématique avec CAS.

MOTS CLÉS: Sémiotique, programme informatique pour l'algèbre, conversions et traitements, signe de Peirce, polynôme de Mac Laurin.

1. Background

In 2005 I introduced the CAS, *Mathematica*, into the Mathematics I Major course at the South African university where I teach mathematics. *Mathematica* is software which transforms the computer into a powerful calculator which the user may use to do symbolic algebra, generate graphs, define functions and so on; it also has many inbuilt mathematical functions. The Mathematics I Major course is a general course designed for students who intend to specialize in the mathematical or physical sciences. The prescribed textbook for the course is the American undergraduate mathematics textbook 'Single Variable Calculus' by J. Stewart (2003).

My hope was that the introduction of CAS would support and enrich mathematical learning. There were many reasons, both mentioned in the literature and from my own experience, why this might happen. For example, the ability of the user to use computer software to move between different representations of mathematical objects may promote conceptual insight (Heid & Blume, 2008, Tall, Smith & Piez, 2008); the use of CAS to reify certain processes into objects could potentially afford a new way of working with mathematical processes (for example, one can use CAS to manipulate a function as if it were an object); the separation of the execution of mathematical tasks by the computer from the planning of mathematical tasks by the learner could result in an increased focus on conceptual planning and problem–solving (Dörfler, 1993).

At the same time research, particularly from France, was beginning to show that the introduction of technology into mathematics classrooms was unexpectedly complex (for example, Artigue, 2002; Trouche, 2005). These researchers argue that the successful introduction of technology involves the development of a bidirectional relationship between user and artefact in which the learner has to construct personal schemes which turn the artefact into an instrument for learning and "the possibilities and constraints of the artefact shape the techniques and conceptual understanding of the user" (Drijvers and Trouche, 2008, p. 368). This process is called instrumental genesis.

With this research about the complexities and promises of technology as a tool for mathematical learning in mind, I decided to monitor what was happening in the computer laboratories of the Mathematics I Major course. My initial observations derived from walking around the laboratories where the students were engaging in CAS–based tasks. These tasks were specially chosen from the textbook (Stewart, 2003) or designed by me. The pedagogical intentions of the tasks were to promote consolidation of mathematical ideas introduced in lectures, or to anticipate new concepts that were about to be addressed during regular lectures (six lectures a week). The students attended one hour sessions fortnightly in the computer laboratories. They were also free to use the computers in their own time.

2. A SEMIOTIC APPROACH

While observing students, I was aware of the well-known epistemological problem: it is impossible to see or know what anyone is thinking. The researcher has access to the person's production and transformation of signs (for example, utterances, algebraic symbols, numbers, or graphs). That is all. Accordingly, the idea of using a semiotic perspective, which looks at the production of signs, became an attractive possibility as a means to an understanding of the students' mathematical activities.

2.1. *What is a Sign?*

C.S. Peirce (1839–1914), one father of modern semiotics[1], proposed that all thinking is performed upon signs of some kind or other, imagined or perceived. He argued that signs are not only a means of signifying or referring to an object; rather they are "means of thought, of understanding, of reasoning and of learning" (Hoffmann, 2005, p. 45). Thus a sign must be experienced meaningfully; it must signify to someone something other than itself. For example, a green traffic light is a sign that tells one to go; it is not there to make one think of greenness. In the phrase '$a = b$', '=' is a sign which tells us that *a* and *b* are equal; it is not there to make us think of the shape '–' or the combination of shapes '–' and '–'.

According to Peirce (1998), all signs have a triadic structure: a representamen (inscription) which refers to the form which the sign takes (not necessarily material), an object (a physical thing or an abstract concept) and an interpretant (the idea or meaning of the object for an individual). In this respect, Peirce's elaborated definition of a sign is.

> anything … which mediates between an object and an interpretant; since it is both determined by the object relatively to the interpretant and determines the interpretant in reference to the object, in such wise as to cause the interpretant to be determined by the object through the mediation of this "sign" (p. 410).

In this article, unless otherwise stated, I assume Peirce's triadic structure of a sign. Examples of mathematical representamen are symbols, words, graphs. Examples of mathematical objects are definitions of the derivative, the function, the rectangle. Examples of interpretants are ideas or interpretations generated in an individual's mind by the representamen of the object. For example, a graph of a parabola with vertex at the origin and domain (–10, 10) is a particular representation (the representamen) of the mathematical object, a quadratic function. Different individuals may construct different interpretants for this representamen (for example, the shape of the parabola, the equation, $y = ax^2 + bx + c$). The role of the interpretant in the making of meaning is crucial. The word 'meaning' is used "to denote the intended interpretant of a symbol" (Peirce, 1998, p.218).

Peirce viewed the "signification and construction of meaning as an ongoing process in which an interpretant of one sign becomes a representamen of another" (Sfard, 2000, p. 45). The interplay of signs leads to the possibility of a process whereby the representamen stands for an object which entails an interpretant and this interpretant in turn becomes the representamen for yet another object and so

[1] In Europe, Saussure (1857–1913) was developing a different version of a theory of signs.

on. This process is called semiosis. In ‘good’ learning, semiosis continues until the learner is able to use the mathematical sign in a way that is meaningful to herself and is commensurate with its use by the relevant mathematical community.

2.2. *Mathematics As a Semiotic System*

During the last decades, a semiotic perspective has been developed and applied to the nature of mathematics and mathematics education by, for example, Rotman (1993), Duval (2001, 2006), Radford (2000, 2006), Otte (2006). Although these semiotic accounts and their derivative versions differ in their formulations, the essential basis of all these explications is the claim that the sign and its meaning form a unity.

Rotman’s (1993) semiotic account of mathematics is well-elaborated. His central argument is that mathematics is a written discourse in which symbols and other inscriptions are manipulated in various ways according to a large and complex set of rules. Crucially, he asserts that mathematical thinking and mathematical objects are co–created and mutually constituted by the human mind in mathematical discourse. In semiotic terms, this means that the mathematical object (the signified, in Rotman’s framework) and the representamen (or signifier) are mutually constitutive. Neither can exist without the other and both evolve with each other.

> No account of mathematical practice that ignores the signifier-driven aspects of that activity can be acceptable. It is simply not plausible – either historically or conceptually – to ignore the way notational systems, structures and assignments of names, syntactical rules, diagrams and modes of representation are constitutive of the very ‘prior’ signifieds they are supposedly describing (ibid., p. 33).

Rotman’s view of mathematics is compatible with the Vygotskian argument that language and symbols are constitutive of meaning, rather than just representative of it: “Thought is not merely translated in words; it comes into existence through them” (Vygotsky, cited in Sfard, 2000, p.45). Rotman’s account provides a useful theoretical structure within which to position my particular arguments although its focus is not on the *learning* of mathematics.

In contrast, Duval (2001, 2006) provides a useful formulation of the learning of mathematics using a semiotic perspective. He argues that signs play several fundamental roles in mathematics: they refer to mathematical objects, they allow one to communicate about mathematics and they are necessary for mathematical processing. Furthermore there are a variety of semiotic representation systems, each with its own possibilities, that are used in mathematical activity. The semiotic representation systems comprise natural language (as used in proofs), the registers of numeric, algebraic and symbolic notations, plane or perspective geometrical figures and Cartesian graphs. Duval argues that mathematical activity is the transformation of one semiotic representation into another in the same or different register (2006, p. 107). I will return to the different types of semiotic representations particularly relevant to mathematical activity with CAS in Section 2.5.

According to Duval, mathematical comprehension involves the capacity to change from one register to another "because one must never confuse an object and its representation" (ibid., p. 7). Duval calls the process of transforming the representation (representamen) of a mathematical object from one register to another, a "conversion". He argues that two representations of the same mathematical object in two different registers do not have the same content – they may denote the same object but different registers make different properties of the object explicit. Duval also claims that transforming a representation within the same register is a process intrinsic to mathematical activity. He calls this transformation a treatment. An example of a treatment is solving an equation given symbolically within the symbolic register. I will illustrate inter– and intra–register transformation of signs later. At this stage, suffice to note that I use the terms representation (Duval) and representamen (Peirce) interchangeably to refer to the concrete instantiation of a sign[2].

Although Duval does not draw directly on Peirce or his triadic structure of a sign, I suggest that Duval's semiotic framework provides a useful elaboration of aspects of Peirce's semiotic framework to the mathematics education domain[3]. In particular, Duval's notion of treatments and conversions, in which one sign is transformed into another in the same or a different register, exemplifies the process of semiosis.

2.3. *Semiotics and Appropriation of Knowledge*

Since this article is in the realm of mathematics education, as opposed to semiotics, one must ask: Where does the appropriation of knowledge fit into a

[2] For an interpretant to function as a representamen, it needs to be articulated (for example, through language, spoken or not, or symbols or visual imagery and so on).

[3] Peirce himself elaborated several aspects of his framework to the mathematical domain (for example, diagrammatic reasoning). See Otte (2006).

semiotic account of mathematics? Peirce's semiotic triad (representamen, object and interpretant), divorced as it is from social practice, context and human interaction is not, on its own, illuminative of cognitive activity (see Radford, 2006, for an elaboration of this idea and an informative critique of Peirce). In contrast Rotman's (1993) perspective locates the roots of knowledge in mathematical discourse which is culturally and historically constituted. This perspective acknowledges that mathematics involves communication and that it is an activity or practice. However Rotman's primary and elaborated concern is with the nature of mathematics as a semiotic activity, not with a theory of learning.

On the contrary, socio–cultural theory (Vygotsky, 1978) is a framework in which the appropriation of knowledge is understood as the product of mediated activity within a social and historical context. The role of the mediator is played by a psychological tool or sign, such as words, graphs, algebra. Vygotsky saw action mediated by signs as the fundamental mechanism which links the external social world to the internal human mental processes and he argued that it is "by mastering semiotically mediated processes and categories in social interaction that human consciousness is formed in the individual" (Wertsch & Stone, 1985, p.166). Vygotsky (1981) called that process by which social processes are transformed into internal processes (through the mechanism of semiotic mediation) 'internalisation'. Implicit in Vygotsky's formulation of internalisation is the idea that social processes are mutated and developed by the individual, not just absorbed in their original form: "It goes without saying that internalisation transforms the process itself and changes its structure and functions" (p. 163). In a related fashion, the interpretant is the transformation by the learner of the representamen into a personally meaningful sign in the mind. As such Peircian semiotics melds well with Vygotskian principles: the external sign (the representamen) is mutated into an internal sign (the interpretant) which itself may become the representamen for yet another interpretant. The object is the scientific concept or a physical object.

Expanding on these principles, the appropriation of knowledge, such as knowledge of mathematical objects, is the outcome of the students' activities with signs. These activities depend, inter alia, on the tools used to generate the signs (such as CAS), the pedagogic processes in the mathematics classroom, the cultural context of the learners (for example, their familiarity with computers), the text with its implicit pedagogical intentions, the institution (for example, the institutionalised attitude to, say, problem–solving) and the particular history of the student. Thus mathematical cognition is a semiotically-mediated activity in a particular historical context which involves the internalization by the learner of culturally and historically sanctioned objects (abstract mathematical ideas).

For example, the activities designed by the teacher or selected from a textbook invoke students to engage in certain mathematical activities, sometimes using a tool such as CAS. As a result of these sign-orientated activities (which are framed by the teacher's pedagogical intentions and situated in a particular social, cultural and historical context) the student is expected to internalise the signs in the form of interpretants; these in turn may lead to further activities and hence further interpretants. Ideally this semiosis continues until the signs become meaningful to the learner and the use of these signs by the learner are consistent with their use in the historically and culturally sanctioned mathematics discourse.

2.4. *CAS and Semiotics*

CAS is a tool that can transform mathematical signs in accordance with the standard rules and procedures of mathematics. As such it may be used to mediate in the construction of mathematical knowledge by the individual. To understand how this happens, it is necessary to consider how its use may enable or constrain the generation of a variety of signs and what the existence of CAS–based representamen may mean for the individual's internalisation of mathematical objects. For example, a user may be able to use CAS to effect a conversion from a symbolic to a graphic representamen, a transformation which the user may not have been able to do using paper and pencil alone. The new CAS–generated representamen may be more epistemologically accessible than other representamen of the object, thus enabling the production of a more useful interpretant. In particular the student may notice important properties of the particular object not previously perceived. Likewise seeing different objects in the same register may help the student discriminate between properties of these different objects.

The word 'may' is used advisedly throughout the previous paragraph: the use of a CAS does not in itself guarantee that a user gains access to more powerful representamen. For example, the learner may be unable to interpret the CAS output or she may not know the correct CAS syntax to generate a representamen. In this regard, Pierce & Stacey (2004) have argued that the precise notation of a CAS may be problematic for students and may act as an impediment to the effective use of the CAS.

The framework of instrumental genesis is well–suited to highlighting the sort of knowledge that a student may need to build an effective relationship with a CAS. An effective relationship requires, inter alia, knowledge of syntax, and knowledge of specific mathematics. For example, to select an appropriate window in which to draw a graph, the student using *Mathematica* has to select an appropriate domain and, sometimes a range (a type of mathematical knowledge not explicitly required when drawing a graph by hand). The selection of appropriate window size has been extensively discussed by researchers within the instrumental genesis framework (see Artigue, 2002; Drijvers and Trouche, 2008).

Another example: To solve an equation where all expressions are algebraic, e.g. $x^2 - 2 = 0$, the student can use Solve (exact answer) or NSolve (numeric approximations). To solve an equation which involves transcendental functions, the user has to use the FindRoot command. The syntax of this command is quite complicated: FindRoot [lhs == rhs, {x, x_0}] where x_0 is a first approximation to the root. For example, FindRoot [Cos[x] == x – 1, {x, 2}] outputs the value of x nearest x = 2, for which Cos $x = x - 1$. Besides this syntactical knowledge, use of the FindRoot command also involves knowledge of how to estimate a first approximation (a type of mathematical knowledge). I return to the FindRoot command in episodes 5 and 9 below.

As the examples above illustrate, doing mathematics with CAS entails a type of hybrid knowledge (mathematical and/or syntactical and/or technical knowledge of how to use a computer) different to that required when doing mathematics with paper and pencil.

Nonetheless the user’s task of generating conversions and treatments may be enabled by the use of CAS. For example the use of CAS may allow the user to generate representamen of mathematical objects before she has any substantial knowledge about the properties of the objects she is representing. This differs from the pencil and paper environment. Duval (2006, p. 124) argues that a conversion of representation requires “the cognitive disassociation of the represented object and the content of the particular semiotic representation through which it has first been introduced and used in teaching”. I suggest that certain forms of conversion in the CAS medium may involve different cognitive processes. Indeed, at one extreme, the user may be able to use CAS to convert an object with which she is completely unfamiliar into a new register. For example, she may use CAS to convert the logarithmic function represented by y = Log x into a graphical representation without having any idea about the properties of the log function. Of course, such a conversion does not guarantee an internalisation of the new object, the logarithmic function. But it may help. For understanding, the student would certainly need to perform further conversions, probably under the guidance of a teacher or textbook – for example, isolating certain properties of the logarithmic function, and describing these properties using language or symbols (a further conversion). It is these further conversions which may require the cognitive dissociations of the mathematical object from its semiotic representations.

2.5. *Analytic Framework*

In order to apply a semiotic framework to the mathematical activity with a CAS, a further refinement of Duval's registers within the semiotic representation systems is required. In particular it is useful to distinguish the symbolic and algebraic registers from one another. I refer to the CAS register in which mathematical rules and algorithms are embodied as the symbolic register. Use of this register requires knowledge of the syntax of the relevant command and possibly some non–algorithmic mathematical knowledge. In contrast, activity in the algebraic register requires the learner to explicitly manipulate symbols and/or execute algorithms according to the rules of mathematics. Activity within the numeric register involves manipulation of numbers (as opposed to symbols).

For example, to solve the equation Cos $x = x - 1$, one may use FindRoot command in the symbolic register (this requires a knowledge of relevant syntax and knowledge of how to estimate a first approximation to the root). Or one may apply Newton's approximation method in the algebraic register (this requires algorithmic knowledge). One can also use trial and error in the numeric register to try to find the roots of Cos $x = x - 1$.

I also distinguish different media, CAS or pencil and paper, from one another. Chandler (2002, p. 232) argues that "signs and codes are always anchored in the material form of a medium – each of which has its own constraints and affordances". (The specifics of the CAS medium differ according to the particular CAS, e.g. *Mathematica*, Derive, etc. But each CAS uses its own representamen which are often different to the representamen used in traditional mathematical notation.) The symbolic register exists in the CAS medium only; the algebraic register and the numeric register may exist in paper and pencil or CAS medium.

Duval's distinction between conversions and treatments, and his argument that it is the transformation of one sign into another that constitutes mathematical activity, is very useful for an analysis of semiotic activity. The conversion of signs between the algebraic register, the symbolic register and the numeric is often a non–trivial exercise, for example the solving of a non–algebraic equation as illustrated above. In the analysis below I show that both conversions and treatments assume a specific role in CAS–based work with particular cognitive implications.

Although I draw heavily on Duval's notion of treatments and conversions as an analytic tool, this article is not intended as a direct application or elaboration of Duval's theory. Indeed, Duval has developed many other theoretical constructs which are not directly related to my arguments in this paper.

Since conversions and treatments using CAS necessarily involve the students' effective use of the CAS, I sometimes invoke the framework of 'instrumental genesis' in order to explain how different levels of instrumentalisation may constrain or enhance semiotic activity. Instrumental genesis has been used my many researchers (for example, Artigue, 2002; Drijvers and Trouche, 2008; Lagrange, 2005) to highlight the intricacy of the process whereby the new technological artefact becomes a functional instrument with which students can do mathematics (instrumentalisation). It is particularly useful when it is used to illuminate the difficulties that the learner may experience when using the technology (for example, see Drijvers, 2000, Drijvers and Gravemeijer, 2005). It has also been used together with Chevellard's Anthropological approach (for example, Hitt and Kieran, 2008) to illustrate how the use of techniques (routines and reasoning) in CAS can stimulate the emergence of conceptual thinking (instrumentation).

My purpose in this article is to focus on the way the learner uses CAS to generate and/or transform signs and the relationship of these signs to the learner's construction of interpretants. To this extent, a semiotic framework is apposite for my project. Nonetheless, since the quality of the student's relationship with the CAS necessarily effects the students' semiotic activity with the CAS, I sometimes refer to the framework of instrumental genesis in my interpretation of the students' activities.

2.6. *Research Question*

In what ways does the use of CAS enable or constrain mathematical activity. In particular, how does the use of CAS promote intra– and inter–register transformations? How do these transformations enable or hinder the construction of appropriate interpretants by the learner?

3. Semiotic analysis of students engaging in a CAS-based task

I use a semiotic analysis to illuminate the type of mathematical activity that the use of a CAS may engender. In particular, I demonstrate how a pair of learners engaging in a mathematical task use various signs (utterances, CAS–based signs, text–based signs) to generate new signs. These new signs (with their new interpretants) permit mathematical activity which eventually leads to an internalisation of well–established mathematical concepts and rules. I frame my analysis in terms of the way in which the use of the CAS enables or constrains conversions and treatments, both core aspects of mathematical activity (see Duval, 2006).

3.1. *The Task*

I look at a vignette in which a pair of learners from the first-year Mathematics Major course which I was lecturing and tutoring in 2007, engaged in a particular task. This task was part of a longer assignment which was given to the students to work on in pairs, near the end of the academic year. It was adapted from a laboratory project in the course textbook (Stewart, 2003, p. 212); its purpose was to introduce students to the concept of the Maclaurin polynomial before the student had been introduced to the concept in regular mathematics lectures. The assignment involved the use of CAS and paper and pencil.

Although I only analyze the students' semiotic productions in one task (Task Four), I contextualize the analysis by briefly describing the tasks which the students had already completed as part of the assignment. In the first task, students used *Mathematica* to sketch a graph of the exponential function $f(x) = e^x$. They were then required to use CAS to generate graphs of the corresponding 3^{rd}, 4^{th} and 7^{th} degree Maclaurin polynomials (whose symbolic expressions were given in the handed–out assignment) on the same set of axes as the original exponential function. The purpose of this task was to give students initial access to the object, a Maclaurin polynomial, through the conversion (by the CAS) of a symbolic representamen to a graphical representamen of specific Maclaurin polynomials.

In the second task, students had to generate, symbolically and graphically, the first order Maclaurin polynomial, L, of $f(x)$ = Cos x given the identities $L^{(n)}(0) = f^{(n)}(0)$, $n = 0,1$. Similarly, in the third task, students had to generate the quadratic approximation (second order Maclaurin polynomial) p, of $f(x)$ = Cos x given that $p^{(n)}(0) = f^{(n)}(0)$, $n = 0,1,2$.

Task four is the following:

Determine the values of x for which the quadratic approximation $p(x)$ found above is accurate to within 0.1. [Hint: Graph the functions, $f(x) = \cos x$, $y = p(x)$, $y = \cos x + 0.1$ and $y = \cos x - 0.1$ on a common screen.]

In order to attempt this task, students need to know, inter alia, that $p(x) = 1 - {}^1/_2\, x^2$. The students in this vignette have found this result in the third task.

3.2. *Data Gathering and Data Presentation*

Five pair of students were audio–taped while doing Tasks Two, Three and Four; their CAS keystrokes were recorded by software (Bulent) and I (the researcher) received printed outputs of their computer work as well as their final assignments. The students were volunteers from the classes that I tutored in the computer laboratories. They came at designated times to my office and I sat in the office throughout their session which was about an hour long.

In the analytic description below, I present parts of a transcript from one taped session (compiled using both the audio–tapes and the keystroke files) involving a pair of students Sipho and Temba. Both Sipho and Temba are average students; at the end of the academic year, they each passed the Mathematics I course with grades of 50% and 53% respectively (the mean mark for the course was 55%).

I have selected excerpts of the original transcript so as to illustrate my arguments. In the transcripts the students' utterances and the commands they entered into the computer are numbered. Where I have omitted utterances or written commands, I use the semicolon character **:**. When the transcript is not particularly informative but is necessary for continuity, I describe the activities that took place rather than reproduce the transcripts. Each episode represents an activity with a specific focus, for example, generating a graph, interpreting a graph and so on.

3.3. *The Vignette*

3.3.1. *Episode 1*

We start with Sipho reading the task out loud (line 1).

[1] Sipho: (reading) "Determine the values of *x* for which the quadratic approximation *p* (*x*) in Task 3 (a) is accurate to within 0.1". In other words, we must say in which interval will approximation of the quadratic give us values that are less than or equal to 0.1… as far as deviation from the actual values goes. Does that make sense?

[2] Temba: Ya. It does. But it's difficult to understand. (Laughs)

[3] Sipho: So like/

[4] Temba: Within 0.1….the notion of 0.1.

[5] Sipho: Ya. Within 0.1. So the value that it gives, it should be within 0.1. It can be less than 0.1 or greater than – 0.1.

[6] Temba: And they give us a hint.

Interpretation: In this episode Sipho attempts to use "other words" (line 1) to transform the set of representamen in the text into a new set of representamen, still in the language register. He uses the phrase "which interval" to refer to the required values of *x* and the phrase "values that are less than or equal to 0.1... as far as deviation from the actual values goes" to refer to the idea given in the original text of "accurate to within 0.1" (line 1). In line 5, he further elaborates the notion of 'accurate to within 0.1' by generating a new representamen: "can be less than 0.1 or greater than – 0.1".

Analysis: The students are trying to find different ways, using language, to denote the same mathematical object to which the text refers. Sipho does this by re–phrasing given descriptions of specific attributes of the object. This creates new representamen, and thus new interpretants. This episode represents a treatment. Although the medium of sign production is changed (from written text to oral text), the students are still working within the language register.

3.3.2. *Episode 2*

Temba and Sipho now plot all four graphs on one screen (Figure 1) as suggested in the Hint. They use domain $(\ 4\ ,4\ )$. As a result all four graphs are very close together in the output; it is consequently difficult to distinguish one graph from another. Despite this the students are able to generate several meaningful signs from the CAS-generated graphs.

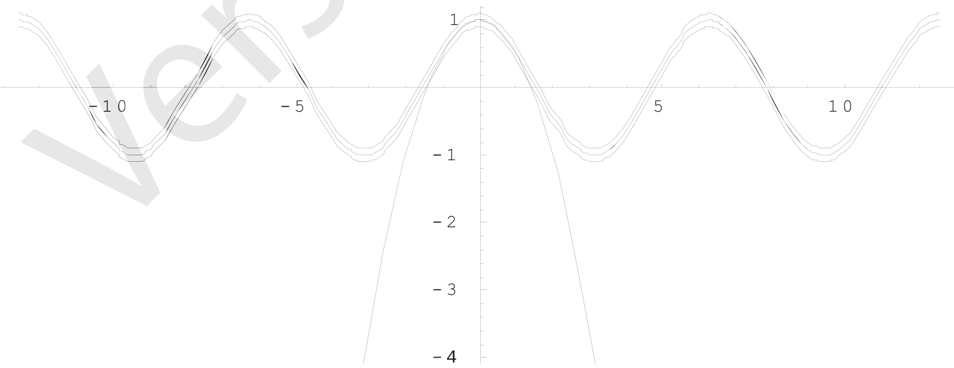

Figure 1. CAS Generated Graph of $Cos\ x, Cos\ x \quad 0.1, Cos\ x \quad 0.1$ and $p(x) \quad 1 \quad \frac{1}{2}x^2$.

[7] Temba: No! What's happening there (referring to screen, ie figure 1).
[8] Sipho: I'm not too sure. Okay, oh ya.
[9] Temba: Oh ya.
[10] Sipho: I can see what is happening. It's shifted in two directions.
[11] Temba: Oh. The centre one. The one in the centre. If you can see. That's probably the Cos one, Cos x. And then minus 1 for the bottom one. Minus 0.1, I mean. And plus 0.1.
[12] Sipho: They are saying: which values of x… its accurate to within 0.1. Wouldn't that be where they intersect? Do you see what I am saying? Like you have this one.
[13] Temba: Um.

Interpretation: Temba and Sipho have used CAS to transform part of the language–based description given in the task (the Hint) into new representamen in the graphical register. Initial interpretation of this CAS–generated representamen (Figure 1) presents its own difficulties (lines 7 – 8) but in line 10, Sipho claims that he "sees what is happening" (the interpretant) which he partially explains (line 10), using the language register. Sipho's explanation, "shifted in two directions", whilst not clear to the outsider, clearly has some value for Sipho. After all, he asks: "Wouldn't that be where they intersect?" (line 12). Meanwhile Temba generates a new interpretant with a new language–based representamen: that is, he explains (line 11) that the centre graph is Cos x, and that the lower graph is Cos $x - 0.1$.

Analysis: In this episode, we see how a use of CAS enables a conversion of the representamen in the language register into the graphical register. Although the students are unable to use CAS to effect an optimum conversion (the graphs in a more appropriate window), they use CAS to gain access to an alternate representation of the mathematical objects (the different functions, the interval of interest) described in the task Later (lines 10 –12) Temba and Sipho move between the graphic (on CAS) and the language registers (both written and spoken) in order to make sense (generate an interpretant) of the objects. This represents the beginning of a conversion; it is a beginning because the language describing the mathematical notion of "which values of x … its accurate within 0.1" in line [12] is not yet disassociated from the graphic representation.

3.3.3. *Episode 3*

Sipho uses paper and pencil to generate yet another representamen, a rough sketch of the four graphs (Figure 2).

[14] Sipho: You have, you have a Cos graph coming like this. And you have Cos plus 0.1 and you have Cos -0.1 (drawing with pencil the four graphs – figure 2). Then you have this quadratic estimates over here.

[15] :

[16] Temba: Okay do you see at this end… I'd say, um. You see where… what will, what will the quadratic do here. Won't it cut the Cos - 0.1 there? And then not go into these graphs. Right? (looking at hand-drawing and screen).

[17] :

[18] Temba: Like what I am trying to say to you is, we must equate our *p* (*x*) to that point there and this point here (darkening points of intersection on figure 2). So it's in between there… the values where it is accurate.

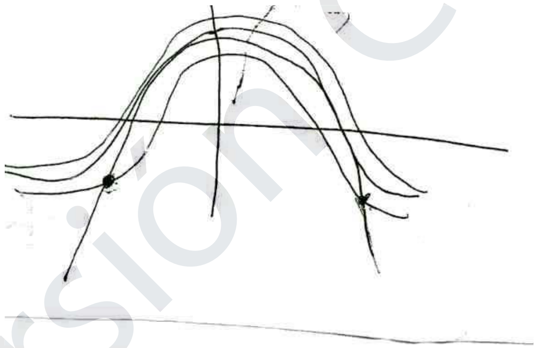

Figure 2. Hand Drawn Graph of $Cos\ x, Cos\ x \quad 0.1, Cos\ x \quad 0.1$ and $p(x) \quad 1 \quad \frac{1}{2}x^2$.

Interpretation: Sipho spontaneously uses paper and pencil to generate yet another graphical sign consisting of the four graphs (line 14). This new graphical sign both depends on the previous representamen (with their interpretants) and looks forward to the generation of future signs. The graphs represent a similar mathematical object to that of figure 1, but with different domain and scale. After further discussion about points of intersection (omitted here), Temba indicates that the *p* (*x*) graph will only cut the Cos *x* – 0.1 graph (line 16). That is, he creates a new interpretant from previous interpretants. A little later (line 18) he is able to transform this sign (the interpretant) into a plan of mathematical activity: “we must equate our *p* (*x*) to that point there and this point here”. Although his statement is somewhat incoherent, it functions as a new sign for him and for Sipho since both of them immediately attempt to solve the equation, Cos x – 0.1 = 1 – ½ x^2 (see following episode).

Analysis: In this episode, we begin to see how Sipho’s hand–drawing of the functions affords the students new insights, allowing them to generate new interpretants about the relationships of the different functions (in particular the relationships between Cos x – 0.1 and 1 – ½ x^2). This illustrates how Duval’s argument (2001) that representations of the same object in different registers make different aspects of the object visible, may be applied to representations of the same object in the same register but in different media, in this case the CAS–generated graph and the hand–drawn graph. As in Episode 1, I regard this as a treatment.

3.3.4. *Episode 4*

Interpretative Description: Sipho tries to use paper and pencil mathematics (trial and error in the numeric register) to solve Cos x – 0.1 = 1 – ½ x^2. However he soon abandons this attempt because “working it out this way is not nice”.

Analysis: Transforming the graphic signs in episodes 1 and 2 into the equation Cos x – 0.1 = 1 – ½ x^2 represents a conversion. However Sipho’s attempt to algebraically solve this equation (a treatment) is unsuccessful; the only way of hand–solving an equation involving a transcendental function and a polynomial is by using a numeric approximation technique, such as the Newton–Raphson method. Such an equation cannot be solved through algebraic manipulations.

3.3.5. *Episode 5*

Meantime Temba is using the *Mathematica* handbook (written specifically for the Mathematics I Major course) to find an appropriate *Mathematica* command to solve this equation. He suggests the use of the FindRoot command (in the Symbolic register). This is apposite since the FindRoot command must be used to solve an equation involving a transcendental function and an algebraic function (see section headed ‘CAS and Semiotics’ for a fuller discussion of this command). Presumably Temba’s knowledge is consequent upon previous class activities in which he used this command.

Temba finds an example of the FindRoot command in the handbook. However, instead of looking at the previous page of the handbook where an explanation of *how to use* the command is given, Temba and Sipho only look at the example. This is: FindRoot $[\mathrm{Cos}[x] == x - 1, \{x, 2\}]$. They are unable to decode the syntax of this command, and they enter the command FindRoot $[h[x] == p[x]]$ into the computer. The syntax is incorrect and they receive an error message – see figure 3.

FindRoot $[p[x] == h[x], x]$
FindRoot :: fdss: Search especification x should be a list with a $2-5$ elements. More...

Figure 3. Error Message

Analysis: In this episode we have an example of an unconsummated transformation. Here the students' attempt to use CAS to effect a treatment (the solution of an algebraic problem) is thwarted by the syntax of the FindRoot command. This episode illustrates how the students' practices (the students' not accessing appropriate resources such as the description together with the example of the FindRoot command) constrain their mathematical activities. It evidences the students' limited instrumentalisation of the CAS.

3.3.6. *Episode 6*

Temba suggests that they use CAS to generate the graphs of Cos $x - 0.1$ and $p(x) = 1 - ½\, x^2$ only. Sipho agrees and suggests the domain $-\pi/4$ to $\pi/4$.

[19] Temba enters Plot command to plot Cos $x - 0.1$ and $p(x)$ on domain $-\pi/4$ to $\pi/4$
[20] A window with graph of Cos $x - 0.1$ and $p(x)$ appears (Figure 4). But there are no visible points of intersection.
[21] Temba: Oh. I did something wrong. Ne?
[22] Sipho: No. Its fine. All it means is that they intersect further down.

Interpretation: Informed by the previously–generated interpretants and their failure to effect a useful conversion of the problem with pencil and paper or CAS, the students attempt to transform their previous graphical representamen into a new graphical representamen. Presumably their goal is to make the relevant information (that is, the points of intersection) more visible. However, the domain is too narrow ($-\pi/4$ to $\pi/4$) and the points of intersection lie outside the domain. See figure 4.

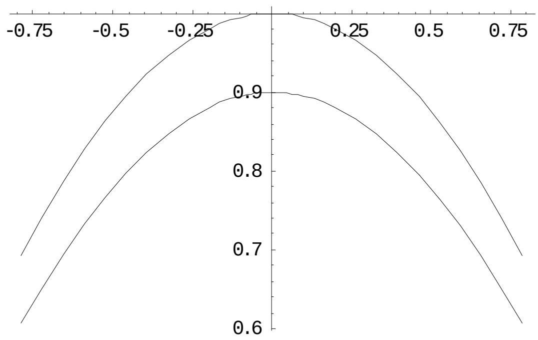

Figure 4. CAS–Generated Graph of Cos $x - 0.1$ and $p(x) = 1 - \frac{1}{2}x^2$; Domain is $(-\pi/4, \pi/4)$.

Temba, realizing that the graphs should intersect (presumably based on previous interpretants) assumes that he "did something wrong" (line 21). But Sipho seems to have a more refined internal picture (interpretant) of what the graphs should look like and he correctly states (line 22) that the graphs intersect at a point(s) outside the domain.

Analysis: In this episode we see how use of CAS to effect a treatment may not be straightforward. To draw the graphs in an appropriate domain, the user of the CAS needs to have mathematical awareness of an appropriate domain (although one of the virtues of CAS is that the graphs can be drawn easily with different domains). More importantly, in order to interpret the CAS–generated graphs (a conversion from graphical register to language or symbolic register), the user needs to have some prior idea of what the graphs should look like (in this case, the graphs must intersect). That is, in order to use the CAS successfully as a tool for conversion or treatment of representation, the user may need to have prior knowledge of the mathematical object she is trying to represent.

3.3.7. *Episode 7*

Temba now re–plots the graph using the domain $-\pi$ to π. A much clearer picture (Figure 5) emerges.

Analysis: Representamen and their interpretants, generated from the previous CAS graphs, together with prior knowledge about Cos *x* and *p* (*x*), enable the users to generate this representation on CAS. This exemplifies a further treatment.

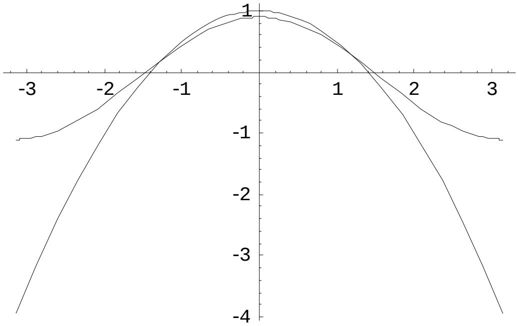

Figure 5. CAS–Generated Graph of Cos $x - 0.1$ and $p(x) = 1 - \frac{1}{2}x^2$; Domain is $(-\pi, \pi)$.

3.3.8. *Episode 8*

Interpretative description: Unable to use a symbolic command such as FindRoot, Temba & Sipho now decide to use trial and error to find value(s) of x where Cos $x - 0.1 = 1 - ½x^2$. Guided by the approximate values of the points of intersection on the CAS graph they substitute numerical values into the CAS–based functions $p(x)$ and Cos $x - 0.1$. This activity is not fruitful since their visual estimates from the graph are not accurate. The activity takes place in the numeric register. Figure 6 gives an example of this activity.

$Cos[1.3] - 0.1$

0.167499

$1 - \frac{1}{2}\left(1.3\right)^2\Big)$

0.155

Figure 6. Numeric Substitutions

Analysis: In this episode, the students try to use the affordances of CAS to find the points of intersection of the two relevant graphs. Specifically, they attempt to generate a new representamen of the points of intersection of $p(x)$ and Cos $x - 0.1$ in the numeric register (a conversion). However working with this new representamen through trial and error (a treatment) is not very useful since it is not guided by any systematic approach.

3.3.9. *Episode 9*

The researcher intervenes. She suggests that the students use FindRoot command (symbolic register). They try to do this but as in episode 5, they use incorrect syntax.

[23] Temba types FindRoot[p[*x*] == h[*x*], *x*].

[24] The computer gives Error message:

FindRoot [*p*[*x*] = = *h*[*x*],*x*]
FindRoot :: fdss: Search especification *x* should be a list with a 2 – 5 elements. More...

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[25] Researcher suggests that they look up the syntax, and they start looking in handbook.

[26] Temba types FindRoot [1 – ½ x^2 – Cos[*x*] + 0.1 = 0, *x* = 0].:

[27] There is a lot of discussion around Syntax. But they are still unable to extrapolate how to use FindRoot sign from the example in the handbook.

[28] Temba now types FindRoot [1 – ½ x^2 == Cos[*x*] – 0.1 = 0, *x* = 0].

[29] He changes syntax to FindRoot [1 – ½ x^2 – Cos[*x*] + 0.1 = 0, {*x*, 0}].

[30] Researcher intervenes. She turns to the previous page in the handbook in which the FindRoot command is explained, rather than just exemplified; she uses this explanation to clarify to the students how to use FindRoot expression.

[31] Temba now changes syntax to FindRoot [1 – ½ x^2 – Cos[*x*] + 0.1 = 0, {*x*, 1.2}].

[32] Researcher shows the students that they have written o, not 0. She also points out that there is only one = sign (there should be two = signs).

[33] Temba types FindRoot [1 – ½ x^2 = = *Cos x* – 0.1,{*x*, 1.2}].

[34] The computer outputs 1.26124.

[35] The one hour is now up and, after some general remarks, the students leave the office.

Interpretation: Temba and Sipho are again severely hindered by poor syntax (lines 23, 26, 28, 29, 31) and they use poor strategies when trying to correct the syntax. For example, Temba (line 26) replaces *p*[*x*] with 1 – ½ x^2 and *h*[*x*] with Cos *x* – 0.1 and moves them both to the left of the equal sign. Also they do not attempt to interpret the Error message (admittedly it is rather opaque – line 24). They are probably tired and demotivated as well, given that at least fifty minutes has passed since the beginning of the session. This is evidenced by their careless use of the alphabetic o rather than numeric 0 (lines 26, 28, 29, 31) and Temba's uncharacteristic poor use of = sign (line 28).

Clearly the example of FindRoot in the text has little semantic value for these students, as evidenced by their use of 0 as a first approximation to *x* (Line 29). (It is a poor first approximation because 0 is precisely at the midpoint of the two roots).

Finally with the assistance of the researcher (lines 25, 30 and 32) they are able to use the FindRoot command correctly; the computer outputs the answer 1.26124 (Figure 7). The students accept this value as the *x*–value of one point of intersection.

FindRoot $[1-\frac{1}{2}x^2 == Cos\ x - 0.1, \{x, 1.2\}]$

$\{x \to 1.26124\}$

Figure 7. Successful Use of FindRoot Command

Analysis: In this episode, as in episode 5, the students attempt to generate a symbolic representation of the relevant (to their task) information contained in the graphic representation of figure 5 (a conversion). However they are stymied both by their lack of knowledge of syntax and their praxis (not 'knowing' how to access information in the manual) although with guidance from the researcher they are finally able to generate the appropriate FindRoot command. Thus we see how the use of CAS to effect a conversion may be hindered by an inability to use syntax correctly. This is yet more evidence of these students limited instrumentalisation of the CAS.

3.3.10. *Episode 10*

Later, outside the research session, they use FindRoot command correctly and they successfully complete the task.

4. Discussion

In the above vignette, students' generation of new signs via treatments (transformations within a register) and conversions (transformations between different registers) in both the CAS and pencil–and–paper media constitute their mathematical activities. That is, mathematical activities are essentially semiotic activities.

Of particular interest is the way in which the use of CAS promotes intra– and inter–register transformations. With regard to conversions, we see how the use of a CAS may afford access to alternate representamen of the objects. For example, in episode 2 the students use CAS to generate a graphic representamen of the four functions referred to in the task. The students are then faced with an epistemological problem: they need to isolate the attributes of the representamen which relate to the given task [4]. In the vignette they do this through generating further representamen, for example, the hand–drawn graph (Episode 3); the CAS–generated graph (Episode 7); the equations in the CAS numeric register (Episode 8); and finally the FindRoot command in episode 9. These new representamen enable students to isolate different attributes of the objects (the four graphs and their relationship to the task) and in this way to enhance their interpretants of the different objects. Semiotically speaking, the students use CAS to enable a process of semiosis (transforming one interpretant into another) and therein lies the value of CAS.

Furthermore, the use of CAS enables access to representamen which would have been unavailable to the student without a CAS and this broadens the type of mathematical activities available to the student. For example, in episode 9, the students use FindRoot command to find the points of intersection of Cos $x - 0.1$ and $1 - ½\, x^2$. Although the students battle to effect this conversion (I return to this point a little later) they are able finally to solve the equation and, in this way, to complete the task (Episode 10).

In a CAS environment, the user may generate new signs (the answer to an algorithmic procedure, the plot of a graph, and so on) with CAS rather than with pencil and paper. I suggest that this outsourcing of computation to the computer has a profound effect on the skills needed to interpret the CAS output (that is, to effect a conversion of signs in the graphic or symbolic registers to the language register). In the semiotic analysis above, we see how the students struggle to effect a conversion of signs in the graphic or symbolic registers to the language register. For example, in episode 2, the students take an inappropriate time to interpret the CAS–generated graphs. However, and as discussed above, their interpretation is ultimately mediated by the generation of further representamen. In this instance, the use of CAS may have promoted interpretation (in terms of providing the user with different representations with which to generate complementary interpretants). Notwithstanding this argument, I suggest that there are instances (not evidenced in my vignette) where signs generated by CAS, and with

[4] The task relates to finding the interval for which the second degree Maclaurin polynomial approximates the cos graph to 0.1 units.

which the student is unfamiliar, may be so opaque as to be indecipherable by the student. This requires further research.

Conversions in the CAS–based environment may also be hindered by the intricacies of the required syntax of the CAS. This is exemplified by the students' frustrated attempts to compose the FindRoot command (Episodes 5 and 9). Problems with syntax may be a result of poor praxis (rather than cognitive difficulties) and/or a limited level of instrumentalisation of the CAS. In these episodes,, the students' difficulties are partly a consequence of 'not knowing how' to access information about this command. That is, the students could have turned to the previous page in the handbook to find more information about how to use the FindRoot command. Or they could have used the Help function in *Mathematica* (both issues of praxis). Related to this, the pedagogical environment should have focused more on strategies to help students instrumentalise the CAS. (This suggests the need for research into appropriate pedagogic strategies for effective instrumentalisation).

Treatments mainly take two forms: transforming graphs into new graphs (through the use of a more apposite domain) and executing algorithms (using the FindRoot command in this vignette). The execution of a particular algorithm is usually extremely easy, provided the task has been correctly transformed into a suitable register (a conversion). But the user is not required to understand the algorithm which the computer uses to solve the equation. See end of episode 9 where students finally use the FindRoot command successfully. However, and as discussed above, a challenge may lie in the interpretation of the output (a conversion).

Treatments involving the transformation of one specific graphic representation of a function into another graphic representation of that function, usually through change of domain, may be cognitively complex. For example, the user may need to already have an idea of what the graph should look like. In episode 6 we see how Sipho recognised the inadequacy of the CAS graphic representamen, presumably because of prior knowledge and/or because of exposure to previous representamen of the graphs (see episodes 2 and 3). As a result the students were able to effect an appropriate treatment of the graphic sign.

Notwithstanding the non–trivial difficulties with the use of CAS, the semiotic analysis illustrates how the use of CAS ultimately affords the students an understanding of which interval they are looking for. Specifically the students use the graphical signs (Figures 1, 2 and 5) to gain crucial insight into which functions need to be equated. They also use CAS to numerically investigate the values at which the quadratic approximation $p(x)$ is within a distance of 0.1 from the Cos graph (Figure 6). Although this numerical approximation is not necessary for solving the problem, it presumably enriches and enhances the students' understanding of the mathematical task and object. However, the students inability to use the appropriate syntax for the FindRoot command severely hinders their mathematical activity in the latter part of the task (Episodes 5 and 9). This strongly suggests that the level of instrumentalisation of the CAS is profoundly related to the semiotic activity of the students.

5. Conclusion

The vignette illustrates how the intra– and inter transformations with CAS may promote semiotic (that is, mathematical) activity. With reference to Vygotsky, the signs which the students generate with CAS or paper and pencil or through utterances, mediate the internalisation process. That is, the students internalise different representamen into various interpretants thereby internalising the outside world. These interpretants are further transformed and mutated by the learner into new representamen with new interpretants and so on. Specifically the different representamen of the same object enable the learners to construct different interpretants for that object. This is revealed by the new signs that they generate in their mathematical activities. In turn, these new signs reveal important properties of the mathematical object under consideration. For example, in the vignette above, different representamen enable the student to construct various interpretants of the relationship of the (quadratic) Maclaurin polynomial, *p* (*x*), to Cos *x* (such as the extent of its approximation in a particular interval, the fact that *p* (*x*) is less or equal to Cos *x* for all $x \neq 0$, and so on). Also, seeing different objects such as the quadratic approximation and Cos *x* in the same register enables the student to discriminate between properties of these different objects (for example, the quadratic approximation is accurate to within 0.1 of the Cos graph for $-1.26 \leq x \leq 1.26$).

We also see that the use of CAS for conversions and treatments is not straightforward. In particular, we see that the construction of the CAS–based signs and interpretation of CAS output may be particularly problematic. With regard to construction of signs (a conversion), difficulties with syntax may drastically limit the usefulness of CAS. In this regard, teachers need to be aware of the importance of an adequate instrumentalisation of the CAS. We saw in the above vignette, how students' semiotic activity was severely limited by their lack of knowledge of how to use the FindRoot command. Interpretation of CAS output (a conversion) may also pose its own unique challenges. Unlike in the paper and pencil environment where the user is always actively involved in constructing the output (e.g. hand–drawing a graph), the CAS user is usually not directly involved in generating the output other than entering an instruction into the computer. Thus interpretation may be a paradoxical endeavour: the user needs to know which properties of the object (say, features of a graph, or roots of an equation) to focus on in order to use these attributes to cognitively construct the object for herself; this may be where pedagogic guidance (via specially designed tasks or discussion) is required. Treatments in the CAS–based environment may also be problematic. In particular treatments involving the transformation of one specific graphic representation of a function into another graphic representation of that function, usually through change of domain, may require prior knowledge of important properties of the function.

A semiotic analysis of different students doing different tasks in a different pedagogical and cultural space would no doubt reveal other possibilities of students' engagement with CAS. Certainly further research focussing on the relationship between the process of instrumental genesis and the evolution of the learner's semiotic activity in a CAS–based context would be very fruitful for further understanding technology's role in the learning of mathematics.

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