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ON EMPIRICAL RESEARCH IN THE FIELD OF USING HISTORY IN MATHEMATICS EDUCATION

RESUMEN. Este artículo aborda la cuestión de la investigación empírica en el campo del uso de la historia en Matemática Educativa. Más precisamente, se enfoca en el papel que la investigación empírica puede tener en la discusión de por qué usar la historia en Matemática Educativa y cómo hacerlo. Esto es ejemplificado principalmente a partir de dos estudios de investigación empírica sobre el uso de la historia en el programa de matemáticas del bachillerato Danés. También se ilustra la manera en que ambos, tanto el diseño como la metodología de investigación de estos dos estudios, dependían del propósito inicial de usar la historia como un objetivo más que como una herramienta. Finalmente, se establecen perspectivas sobre los posibles beneficios de incrementar la cantidad de investigación empírica hecha dentro del campo del uso de la historia en Matemática Educativa.

PALABRAS CLAVE: Historia en matemática educativa, historia como un objetivo, historia como una herramienta, investigación empírica, diseño y metodología de investigación, creencias de los estudiantes e imágenes de las matemáticas.

ABSTRACT. This paper addresses the question of empirical research in the field of using history in Mathematics Education. More precisely, it focuses on what role empirical research may serve in the discussion of why to use history in mathematics education and how to do it. This is exemplified mainly by referring to two empirical research studies on the use of history in the Danish upper secondary mathematics program. Also it is illustrated how both the research design and research methodology of these two studies were dependant on the initial purpose of using history being concerned with history as a goal rather than history as a tool. Finally, perspectives are drawn on the possible benefits of increasing the amount of empirical research being done within the field of using history in mathematics education.

KEY WORDS: History in mathematics education; history as a goal; history as a tool; empirical research; research design and methodology; students' beliefs and images of mathematics.

RESUMO. Este artigo aborda a questão da investigação empírica no campo do uso da história em Educação Matemática. Mais precisamente, foca-se no papel que a investigação empírica pode ter na discussão sobre como usar a história em Educação Matemática e o modo de o realizar. Isto é exemplificado, principalmente, a partir dos estudos de investigação empírica sobre o uso da história no programa de Matemática do "bachillerato Danés". Também se ilustra a forma como ambos, quer o desenho quer a metodologia de investigação destes estudos, dependiam do propósito inicial de usar a história como um objectivo mais do que como uma ferramenta. Finalmente, estabelecem-se perspectivas sobre os possíveis benefícios de aumentar

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a o número de investigações empíricas realizadas no campo do uso da história em Educação Matemática.

PALAVRAS CHAVE: História em educação matemática, história como um objectivo, história como uma ferramenta, investigação empírica, desenho e metodologia de investigação, crenças dos alunos e concepções sobre a Matemática.

RÉSUMÉ. Cet article aborde la question de la recherche empirique en ce qui concerne l'utilisation de l'histoire dans didactique des mathématiques. Pour être plus précis, l'article a pour sujet principal la fonction que peut prendre la recherche empirique dans le fait de savoir pourquoi et comment on peut utiliser l'histoire dans l'enseignement des mathématiques. Les principaux exemples retenus pour répondre à ces questions sont constitués par deux travaux de recherche empirique portant sur l'utilisation de l'histoire dans le programme de mathématiques pour la préparation du baccalauréat au Danemark. L'article montre aussi que le fait d'opter dès le début pour une utilisation de l'histoire, non pas simplement en tant qu'outil pédagogique mais surtout comme objectif en soi, modifie aussi bien la forme que la méthodologie de recherche utilisées dans ces deux travaux. Pour finir, l'auteur s'interroge sur les possibles avantages d'une recherche empirique plus importante dans le domaine de l'utilisation de l'histoire pour enseigner les mathématiques.

MOTS CLÉS : Histoire dans didactique des mathématiques, histoire en tant qu'objectif, histoire en tant qu'outil pédagogique, recherche empirique, conception et méthodologie de la recherche, idées reçues des étudiants et image des mathématiques.

1. INTRODUCTION

A vast amount of literature is available today on the use of history in mathematics education, e.g. in journals such as *Educational Studies in Mathematics* (ESM), *For the Learning of Mathematics* (FLM), and *Zentralblatt für Didaktik der Mathematik* (ZDM), as well as specially dedicated books on the subject, the most comprehensive one being the 10th ICMI-Study on *History in Mathematics Education* (Fauvel and van Maanen, 2000). Only a very little part of this literature deals with actual empirical research on the use of history of mathematics in mathematics education (for an exemplification of this see e.g. Jankvist, 2007). The literature does, however, offer a variety of arguments on why and how to use history in mathematics education. However, these arguments often seem to be based on the authors' personal teaching experiences or speculations about possible benefits on the students' behalf, and only seldom are empirical data provided to support the claims made. Gulikers and Blom (2001, p. 223) refer to the publications on history in mathematics education as being 'anecdotic', and state that "it is unclear whether and how the (generally positive) experiences can be transferred to other teachers, classes and types of schools." At the previous *Topic Study Group* (TSG) meeting on history at the ICME10 conference in Copenhagen a similar statement was made by Siu and Tzanakis (2004, p. 3) who concluded that "it became clear that enough has been said on a 'propagandistic' level, that rhetoric has served its purpose" and hence argue that what is needed now are empirical investigations on the effectiveness of using history. But what is to be understood by 'effectiveness'? Well, if one accepts a distinction between two fundamentally different purposes of using history in mathematics education, then the 'effectiveness' may concern either of these purposes.

1.1. *Sketching a framework for discussing the 'whys' and 'hows'*

The first purpose, which I shall refer to as *history as a tool*, concerns the use of history as an assisting means, or an *aid*, in the learning of mathematics (mathematical concepts, theories, and so forth). Of course, in this sense, history may be an aid in terms of motivation or affection as well as in terms of cognition. In the second purpose, which I shall refer to as *history as a goal*, history does not serve the primary purpose of being an aid, but rather that of being an *aim* in itself. By this I mean posing and suggesting answers to questions about the evolution and development of mathematics, for instance, about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics and its history (Niss, 2001, p. 10). Of course, the use of history as a goal may have the side effect of assisting the learning of mathematics, but the important thing is that it here is not the primary purpose of using history. In short, one might say that history as a goal concerns the teaching of meta-perspective issues, or *meta-issues*, of mathematics, whereas history as a tool concerns the teaching and learning of the inner issues, or *in-issues*, of mathematics. Thus, when talking about the effectiveness of using history in mathematics education, it seems to me a reasonable approach to distinguish between the effectiveness of history as a tool and the effectiveness of history as a goal.

By the *'whys'* of using history I am referring to the many arguments for actually using and/or integrating the history of mathematics in mathematics education. A display of many of these arguments may be found in Tzanakis and Arcavi (2000, pp. 202-207), where a classification of these into five different classes is given also. However, these five classes of 'whys' may be reorganized into the shorter classification of history as a tool and history as a goal (for an exemplification of this see Jankvist, 2009a). The discussion of the ‘whys’ is strongly connected to the discussion of the *‘hows’*, that is the different ways of actually using and/or integrating the history of mathematics. Concerning this discussion, Tzanakis and Arcavi (2000, p. 208) mention three different main approaches to the integration of history in mathematics education: (1) “Learning *history*, by the provision of direct historical information”, (2) “Learning *mathematical topics*, by following a teaching and learning approach inspired by history”, and (3) “Developing *deeper awareness*, both of mathematics itself and of the social and cultural contexts in which mathematics has been done”. The approaches in 3 seem, in large, to deal with realizing history as a goal, whereas 2 clearly concerns history as a tool. As examples of approaches which belong to 2 we find, for instance, the genetic method by Toeplitz (1927), guided reinvention by Freudenthal (1991), and the use of history in identifying epistemological obstacles for later design of didactical situations by Brousseau (1997). Approach 1 on learning history seems mainly to concern history as a goal, but may perhaps also deal with history as a tool since Tzanakis and Arcavi mention history of conceptual developments.

Thus, in order to ‘measure’ the effectiveness of using history in mathematics education it seems quite clear that one has to consider both the initial purpose (the ‘whys’) for actually using history as well as the way in which history is brought into play (the ‘hows’)[1].

1.2. *Narrowing down the focus of the paper*

Some relevant questions to ask concerning empirical research in the field of using history in mathematics education are, amongst other, what the role of empirical research studies in the discussion of ‘whys’ and ‘hows’ of using history in mathematics education is, and how the research design and the research methodology in such studies depend on the original purpose of the study being concerned with either history as a tool or history as a goal.

As a way of contributing to the answering of these questions I shall exemplify the role of empirical research in the discussion of the ‘whys’ and ‘hows’ of using history in mathematics education by referring to two empirical studies on using history in the Danish upper secondary mathematics program.

[1] A further development of the above framework and a discussion on the relationship between the ‘whys’ and ‘hows’ may be found in the Jankvist (2009a).

The purpose of these studies was history as a goal, and the way of bringing the history of mathematics into class was by carrying out two teaching modules concerned with two different cases of the history of mathematics. By means of these examples, I shall try to illustrate how empirical research may contribute to the discussion of 'whys' (and to some extent also the 'hows') in ways that speculations and 'armchair research', excellent as it may be, cannot. Also, I shall further illustrate how both the research design and the research methodology on these two studies were dependant of the purpose being history as a goal rather than history as a tool. But first let us see what role the history of mathematics is supposed to play at the Danish upper secondary level and which further questions this gives rise to.

2. THE DANISH UPPER SECONDARY MATHEMATICS PROGRAM

At the Danish upper secondary level the students now are to "demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution", knowledge acquired, for example, through teaching modules on history of mathematics (Undervisningsministeriet, 2007)[2]. In such teaching modules the upper secondary mathematics teachers are free to choose the curriculum themselves – something which raises a lot of opportunities when it comes to the inclusion of elements of the history of mathematics.

The official regulations regarding history for the Danish upper secondary mathematics program from 2007 originates from Niss (1980), and the rhetoric is to some extent based on the Danish report on *Competencies and Learning of Mathematics* (Niss and Jensen, 2002), the so-called KOM-report[3], where it says:

> In the teaching of mathematics at the upper secondary level, the students must acquire knowledge about the historical evolution within selected areas of the mathematics which is part of the level in question. The central forces in the historical evolution must be discussed including the influence from different areas of application. Through this the students must develop knowledge and understanding of mathematics as being

[2] All quotes in this section are my own translations from Danish.
[3] Presently there is no English version of the KOM-report.

> created by human beings and, in fact, having undergone an historical evolution – and not just being something which has always been or suddenly arisen out of thin air. (Niss and Jensen, 2002, p. 268)

The focus of integrating the history of mathematics in mathematics education is also discussed in the KOM-report:

> [This] must not be confused with knowledge of 'the history of mathematics' viewed as an independent subject. Focus is on the very circumstance that mathematics has evolved, in environments conditioned by both culture and society, and on the driving forces and mechanisms which are responsible for this evolution. On the other hand, it is obvious that for overview and judgment concerning this evolution to have solidness they must rest on concrete examples from the history of mathematics. (Niss and Jensen, 2002, p. 68)

The talk of "solidness" in the above quote suggests that the meta-issues concerning the evolution and development of mathematics must somehow be *anchored* in the related in-issues of the teaching module. Also, it is quite clear from the quotes above that the focus is on history as a goal rather than history as a tool.

2.1. *Research questions of the two empirical studies*

But how do we even know if students at upper secondary level are at all capable of reflecting on such meta-issues of the evolution and development of mathematics? And if so, then in what ways? For instance, is it possible to anchor the students' meta-issue reflections in the taught and learned in-issues of the teaching module? And from a design perspective, how may such an anchoring be ensured? Furthermore, how do you develop such an 'overview and judgment' of which the KOM-report talks? Or phrased differently, how do you develop the students' beliefs about and images of mathematics to be more coherent and more reflected?

These were some of the research questions of the empirical studies carried out on the 'effectiveness' of using history as a goal – studies which I shall describe in the following sections.

3. CHOOSING CASES FOR THE TEACHING MODULES

As any kind of empirical research may be divided into tasks of design, implementation, and evaluation this, of course, goes for empirical research studies in the field of using history in mathematics education as well. However, when designing historical teaching modules a lot of work goes ahead of the actual design, namely in the selection of suitable historical cases. And in the case of the Danish upper secondary level the task of selecting cases is, perhaps, even more considerable since the historical elements are not dictated by the curriculum.

The KOM-report's requirement that the presentation of the central forces in the historical evolution also should include a discussion of the influence from different areas of application led me to look into the history of more recent and applied mathematics. Though literature in the field of using history in mathematics education is rich on ideas and suggestions as to what elements of the history of mathematics to include on given levels of education, the vast majority of these examples seem to concern the old, or often antique, history of mathematics. This may, of course, not be so strange since the old mathematics often is tighter connected to the mathematical topics of the school curriculum. However, concerning inspiration on what modern history of mathematics or modern applications of (possibly old) mathematics to include in mathematics education not much help was available[4].

3.1. *'General topics' in the History of Mathematics*

One criteria for choosing a specific case, or "concrete example" as said in the KOM-report, was to consider cases which possessed more general features –or 'general topics'– of the history of mathematics as such. By this I am referring to topics or approaches to the history of mathematics which are not case-specific and therefore not restricted to the concrete examples in question. By identifying concrete cases addressing such general topics it would be possible to teach the students something about the evolution and development of mathematics in general by means of a single historical case. But allow me to exemplify what I mean by 'general topics'.

[4] For an elaborated discussion on the benefits of choosing cases from the history of modern applied mathematics, see Jankvist (2009b).

One topic in the history of mathematics and in the history of science in general, is that of *multiple discoveries* (or inventions). There are several examples of this in the history of mathematics, where two persons, or groups of people, have published similar results independently of each other, sometimes at approximately the same time in history – then known as simultaneous discoveries/inventions. One of the more famous examples of this is, of course, the simultaneous developments of infinitesimal calculus by Leibniz and Newton.

Another topic could be that of approaching the development of mathematics though the notion of *epistemic objects* and *epistemic techniques* as introduced by Rheinberger (1997) and modified by Epple (2000). An epistemic object is the mathematical object under investigation and development in a given situation at a given time. The epistemic techniques are the mathematical tools used to carry out the investigation of the object. These techniques can be already well-established mathematical concepts, theories, methods, etc., or they can be developed in the process of the investigation of the object. (I shall exemplify this in subsection 4.2.) The notion of epistemic objects and techniques is a useful working tool for micro-historical approaches to the history of mathematics since they are bound in time and space, and since they are "constructed to distinguish between how problem-generating and answer-generating elements of particular research episodes function, interact, and change in the course of the work of a specific mathematician or group of mathematicians" (Kjeldsen, 2008). Worth noticing is that objects and techniques may shift places and roles in different *epistemic configurations*, i.e. what is the technique in one configuration may be the object of study in another or vice versa. For instance, examples of complex numbers were used as techniques to solve algebraic equations in the 1500s, but later complex numbers became the object of study themselves.

A third general topic is that mentioned in the KOM-report of the driving forces and mechanisms responsible of the evolution and development of mathematics. When discussing such matters one may differ between the *inner* and the *outer driving forces*[5]. Inner driving forces are the forces and mechanisms which are responsible for developing the mathematics from the inside. These can be, for example, intriguing questions, unsolved puzzles and riddles, unproven conjectures, etc. which drive mathematicians in their research. Outer driving forces are the forces and mechanisms which influence the discipline of

[5] The discussion of inner and outer driving forces has some resemblance to the discussion of internalism and externalism.

mathematics from the outside. Examples are societal needs, money, and, not least, war to mention a few. For example, the U.S. funding of research after World War 2 and during the Cold War was a major (outer) driving force for several scientific and technological disciplines at the time.

3.2. *Identifying two cases that address 'general topics'*

Coming up with exemplary historical cases of modern applied mathematics addressing some of these general topics is one thing. Of equal importance is the fact that the mathematics of the cases must be explainable to the students. But how did I then come up with a couple of cases fulfilling these two criteria?

With the birth of the computer era in the twentieth century mathematicians found new ways to apply elements of discrete mathematics both in creating new mathematical disciplines and in solving various 'computational' problems. This together with the fact that some elements of discrete mathematics stand a fair chance of being communicated to students in, at least, upper secondary school suggest that the history of discrete mathematics may be a place to look for relevant cases of the history of modern applied mathematics to be used in mathematics education. Already being familiar with the disciplines, and some of the history, of what may be referred to as mathematical coding (error correction, data compression, and cryptography) this was a natural place to look. I remembered that I a few years earlier while studying error correcting codes and their history had been able to explain the idea of some of these codes as well as several of the related basic concepts in a fairly easy manner to some of my non-mathematician friends. Especially binary Hamming codes can be explained somewhat easily to 'outsiders', and this without even introducing much linear algebra. I thus settled for the early history of error correcting codes (works of Shannon, Hamming, and Golay) to be the topic of the first teaching module.

The idea for the second teaching module sprang out of my experiences from teaching a course in *Discrete Mathematics and Its Applications* to undergraduates at Roskilde University. Following the book by Rosen (2003) of the same name, the students were here introduced to elementary number theory and its application in RSA cryptography. Since some of these students were almost fresh out of upper secondary school it was natural for me to think that this possibly could be taught to upper secondary students as well. Furthermore, I knew that the history of RSA, as well as that of error correcting codes, touched upon several of such ‘general topics’ as discussed above. In the following I shall provide a brief outline of these two historical cases.

3.3. *Case 1. The early history of error correcting codes*

Shannon was employed at the Bell Laboratories when he developed his mathematical theory of communication (Shannon, 1948). As part of this theory, he proved that ‘good and efficient’ error correcting codes exist. Shannon’s proof of this was a non-constructive existence proof. However, he was able to provide one example. This example was due to the mathematician Hamming who was also employed at the Bell Labs. Hamming used the super computers at Bell Labs, but he did not have first priority of these computers, so many of his calculations were run over the weekend. At this point in time computers were only using error detecting codes, meaning that whenever a computer detected an error due to ‘noise’ it would drop the current calculation and move on to the next in line. After finding his calculations dropped two weekends in a row Hamming said to himself: “Damn it, if the machine can detect an error why can’t it locate the position of the error and correct it?” (Thompson, 1983, p. 17) Continuing this line of thought Hamming began developing his error correcting codes and, supposedly, by some time in 1947 he was able to provide Shannon with the example of one of his codes, the today so-called Hamming (7,4)-code (each codeword in the code consists of seven symbols, four of these being information symbols). When the Bell Labs learned about Hamming’s codes they wanted to patent them. This led to a long delay of Hamming’s publication of the codes. Hamming was not able to publish anything on the topic until 1950 (Hamming, 1950). At this time another mathematician, Golay, had already deduced the rest of the Hamming codes from the (7,4)-code provided in Shannon’s 1948-paper, and he even provided a few additional codes of his own, the two so-called Golay codes, and published it all in 1949 (Golay, 1949). This has led to an, today even, on-going dispute of who actually should receive credit for the family of Hamming-codes: Hamming or Golay?

3.4. *Case 2. Public-key cryptography, RSA, and Number Theory*

The public-key cryptography algorithm known as RSA is another example of a bunch of mathematical formulas which were patented. In 1976 the two Stanford researchers Diffie and Hellman published their idea for a public-key cryptography system, thus solving one of the oldest problems in cryptography:

the key-distribution problem (Diffie and Hellman, 1976). However, Diffie and Hellman only knew that the system would work in theory; to make it work in practice they were lacking a so-called mathematical one way function which fitted the description of the system. This function was provided the following year by computer scientists Rivest and Shamir and the mathematician Adleman, all from MIT (the name RSA is due to their initials) (Rivest, Shamir, and Adleman, 1978). The function was based on the fact that it is easy to multiply two large prime numbers together to obtain a large integer, but going the other way, that is prime factoring this integer, is an extremely time-consuming process, making it 'for all practical purposes impossible', which is what defines such a one way function. Previously, prime numbers had been considered to be a branch of mathematics which had no practical applications, a view which, for instance, is ascribed to the mathematician G. H. Hardy (Wells, 2005, p. 120). In producing their algorithm Rivest, Shamir, and Adleman actually relied on various results from number theory. Especially three historical results stand out in the proof of the correctness of the algorithm. These are: the Chinese remainder theorem which is ascribed to Sun Zi (around year 400); Fermat's little theorem from 1640; and Euler's generalization of this, Euler's theorem, from 1735-36. A special twist to the history of public-key cryptography and RSA is that a more or less parallel development took place within the British Government Communication Headquarters (GCHQ) – this one being classified, of course. In the 1960s Ellis, a top cryptographer of GCHQ, had arrived at essentially the same solution as Diffie was to arrive at six years later. But like Diffie and Hellman, Ellis could not find the one way function to fit the description. Not until four years later, in 1973, when a young mathematician by the name of Cocks was employed at the GCHQ was such a function found. Cocks, being familiar with number theory, quickly devised the RSA algorithm, four years before Rivest got the idea for it. GCHQ being a governmental agency kept all this to themselves and this even though the university researchers took patents, created businesses, and earned millions and millions of dollars. Not before 1997 did GCHQ make the story public (Singh, 1999).

4. Research design

The design of the teaching modules consisted of both the design of the teaching material (Jankvist, 2008c and Jankvist, 2008e) as well as the activities to be carried out in class during the implementations.

4.1. *Teaching materials*

In the teaching material for the teaching module on early error correcting codes the students were given an introductory example of two people sending text messages to each other over their mobile phones in order to set the scene for error correcting codes. Also they were introduced to binary numbers and binary representation in general. The selected theory of error correcting codes was presented to them in modern notation along with the history of both error correcting codes, i.e. their 'birth' and 'childhood years', and the already established mathematical concepts and theories on which error correcting codes were first founded. Applications of error correcting codes, both the ones in question and others, were also discussed along the way. As a service for the students, the text was set in two different fonts; one for in-issues, that is mathematics, *and one for meta-issues: history, application, etc.*

The idea of using two different fonts as well as presenting the mathematics in modern notation was repeated in the material for the second teaching module on RSA cryptography. The introductory example here was that of Caesar cryptography (substituting each letter in a message with the letter three places ahead in the alphabet) which led to a discussion of the key-distribution problem, a presentation of Diffie's and Hellman's idea for public-key cryptography, and the use of a mathematical one way function. Hereafter the students were introduced to a selection of elementary number theory (prime numbers, Euclid's algorithm, the fundamental theorem of arithmetic, etc.) and from there moved on to a presentation of congruence and the historical theorems used to prove the correctness of the RSA algorithm.

In both sets of teaching material the mathematics and history were unfolded in parallel; depending on the level of difficulty either in the form of 'stories' or by showing the students translated extracts from the original sources. For example, in the material for the second module the works of Riemann and Hardy were introduced in the form of stories whereas translated extracts were shown from, for instance, Euclid's *Elements* and *Sunzi suanjing*. The materials were designed in order to try to meet the intentions of history as a goal, as described in section 2, e.g. by showing the students that the development of mathematics draws on inner as well as outer driving forces.

4.2. *Essay assignments*

Besides a number of mathematical exercises, or problems, the teaching materials also contained a number of so-called *essay assignments*. Especially, each module contained one large final essay assignment which the students were to do in groups and turn in (in groups as well). The idea of these essay assignments was to force the students (and the teacher) to work with the meta-issues of the history of mathematics in question. Each final essay assignment consisted of a main assignment and a number of supportive assignments. The main assignment, in both modules, was to provide two different accounts of the given historical cases; one focusing on *when* what happened and *who* made it happen, and another focusing on *why* a certain development took place and *how*. In accounting for these two different lines of history the students were to rely on the supportive essay assignments. The foci of these assignments differed between the two modules, a topic which I shall return to in section 5.

In the first module there were three supportive essay assignments: (1) The first was an exercise in reading original texts where the students were provided with an extract from Shannon's paper containing the example of the (7,4)-code and its decoding. Based on this, they were to discuss why this presentation was, in fact, equivalent to the slightly different one that they had been exposed to in the teaching material. (2) In the second assignment they were to identify *objects* and *techniques* in the early history of error correcting codes. In Hamming's work the error correcting codes were the objects under investigation, and the already established pieces of mathematics used to investigate and develop these were the techniques. These techniques included, amongst others, the concept of metric due to Fréchet (1906) as well as elements of linear algebra which may be ascribed to Grassmann (1844). And then the students were to describe what purposes these techniques served in the development of the codes. (3) In the third assignment the students were asked who should receive credit for the codes as well as why they thought mathematicians and historians spend so much energy on clarifying such issues.

In the second module there were also three supportive essay assignments: (1) The first consisted of a reading of about two thirds of G. H. Hardy's *A Mathematicians Apology* (the English edition). Based on this, the students were to account for Hardy's views on pure and applied mathematics and relate them to the history of RSA as well as the need for basic research. (2) The second assignment was on inner and outer driving forces. Based on the teaching material, the students were to see what they could make of the personal motivations of, on the one hand, the newer researchers involved in public-key cryptography (Diffie, Hellman, and Merkle; Rivest, Shamir, and Adleman; Ellis, Cocks, and Williamson) and, on the other hand, the older mathematicians involved in number theory (Euclid, Sun Zi, Fermat, Euler, Gauss, Riemann, and Hardy). These personal motivations, as far as it was possible to say something on these, were then to be related to inner and outer driving forces in cryptography and number theory as a whole. And finally, the newer history of cryptography and the older history of number theory were to be compared and discussed in terms of inner and outer driving forces. (3) The third supportive assignment was, as in the first module, on the question of multiple discoveries/inventions and the attribution of credit. But this time, as indicated earlier, the discussion was to be angled somewhat differently since the students were to take into account the classification of research in governmental agencies.

4.3. *Implementation*

A very important part of the implementation of the teaching modules was the setting of the scene for the essay assignments, both in the teaching materials, as accounted for above, but also in the activities of the modules. The actual implementations of the modules went for about fifteen ninety-minute lessons. The class, in the end consisting of 23 students, was taught by their regular upper secondary mathematics teacher. The module on error correcting codes was carried out in the spring of 2007 during the students' second year of upper secondary school (age 17-18). The module on RSA was carried out in the same class with the same teacher in the winter of 2007 during the students' third and last year (age 18-19). As said, the final essay assignments were done at the end of the modules. The students were then divided into six groups of three to five persons in which they were to do the assignments[6]. Part of the work of the students was carried out in class where they had the opportunity to discuss amongst themselves and get assistance from their teacher.

[6] Some alterations of the groups were made between the modules, but I shall not enter into lengthy discussions of this, since it does not directly affect the data to be displayed and discussed in this paper.

5. RESEARCH METHODOLOGY

The methods used to answer the research questions of the empirical studies (cf. subsection 2.1) encompassed a gathering of data from various sources: data on which to perform different triangulations. The setting of the research design consisting of two modules instead of just one also had some methodological consequences which will be discussed in this section.

5.1. *Data sources*

During the implementation several sources of data were gathered: The students' written essay assignments were collected; one (focus-) group was followed and video filmed during both modules, when doing the mathematical exercises and when working on the essay assignments both; the teacher was also video filmed whenever she lectured on the teaching material; and any mathematical exercises which the students were to turn in were collected. Also the students were given questionnaires before, in between, and after the modules. The questions in these were a combination of questions on their beliefs about, images of, and possibly their attitude towards mathematics, of course mostly concerned with aspects of the history of mathematics, and small test questions on the meta-issues as well as the in-issues of the teaching modules. (Examples of the questions from the first questionnaire will be shown below.) Furthermore, all students in the focus group, as well as representatives for the other groups, were interviewed after turning in the essay assignments and answering the questionnaires. Also the teacher was interviewed before, in between, and after the modules.

5.2. *Selection of focus group students*

The focus group consisted of five students whom were, as well as the other interviewees, mainly chosen based on their answering of the first questionnaire. In total there were twenty questions in the first questionnaire, examples of which are:

- Do you believe it to be important to learn mathematics? Why or why not?
- Can you mention anywhere in your everyday life or elsewhere in

society where mathematics is applied (indirectly or directly)?

- Do you think that mathematics has a greater or lesser impact on society today than one hundred years ago?
- How, when, and why do you imagine that the mathematics in your textbooks came into being?
- Do you think parts of mathematics can become obsolete? If yes, then in what manner?
- Were the negative numbers discovered or invented? Why?
- Do you generally believe that mathematics is something which you discover or invent?
- Is mathematics a science (scientific discipline)? If yes, about what? If no, what is it then?[7]
- What do you think a researcher in mathematics does? What does the research consist in?

As may be observed, the questions were a combination of more historical and developmental questions, sociologically oriented questions, and some philosophical or epistemological questions[8]. Along with these questions concerning their beliefs about or images of mathematics, the students were also asked what they liked the most about mathematics, what they liked the least about mathematics, and if they found themselves good at mathematics, i.e. questions relating more to their attitudes towards the subject. Also the students were asked if the history of mathematics was something they did, or believed that they could, find interesting. Based on the answering of the questionnaires, twelve students were chosen for interviews. These students were chosen so that they represented the views and beliefs of the class in general as best was possible. In the interviews, the students were asked to explain or

[7] The Danish the word *videnskab* (science) is, as the German word *Wissenschaft*, much more inclusive than the English word *science*, and refers to practically all forms of systematic knowledge about nature (the natural sciences), culture (the Humanities), and society (the social sciences), where *science* generally only covers the sciences which use the scientific method to study nature (as opposed to, for instance, the social sciences which may use the scientific method to study human behaviour). However, since there is no overarching English term synonymous to *videnskab*, I have still used the word *science* when translating.

[8] For an analysis of the students' answers to some of the above questions, see Jankvist, 2009c.

deepen their questionnaire answers. From the twelve interviewees a further selection of five students was made, the five students to make up the focus group to be video filmed during the implementations. Also this group should represent the viewpoints of the class as best was possible. For example, the group should consist of students who were skeptic towards the history of mathematics as well as students who found, or believed they would find, it interesting. Also the students should be able to function and work together in the group. In this matter I had to rely on the teacher of the class to help me with her knowledge about and prior experiences with the students. The teacher formed the remaining five groups herself, the only constraint being that there should be at least one interviewee in each of the groups, so that I had the possibility of posing questions to the group's hand-in essays in the interviews.

5.3. *Triangulations*

Due to the focus of the studies being history as a goal some of the above mentioned data sources, and possible *triangulations* between these, were to be considered more important than others. The most essential, or primary, data were the written essay assignments, the video of the focus group doing these, and the questions and answers from interviews and questionnaires concerning the meta-issues and any anchoring of these in the in-issues. Less essential data in this sense were the videos of the teacher, the students' mathematical exercises, as well as the videos of the students doing these. Had the focus of the studies been history as a tool, these would have been the important data. But when the focus was history as a goal they were to be considered secondary in the following sense: If, for instance, a misconception or something otherwise strange or notable was spotted in the primary data sources concerning the essay assignments, then an explanation may be sought for in the secondary data sources. Maybe the teacher made an error in an explanation on the blackboard; something which might explain why some students' misconceptions appear in the essay assignments as well as in the mathematical exercises. In such situations triangulations between the primary and the secondary data sources could be made.

5.4. *Why two teaching modules?*

The study carried out was not a traditional design research study in the sense of a cyclic process of doing thought experiments and instruction experiments (e.g. Gravemeijer and Cobb, 2006), or put another way: designing, implementing, evaluating, redesigning, re-implementing with another population, and so on. This study consisted of a design with one case, an implementation, an evaluation, a design with another case, an implementation with the same population, and an evaluation. Still some 'redesign' was possible from case to case, though not in the cyclic design research sense. This 'redesign' consisted in varying some parameters from the first module to the second, especially in the essay assignments.

One parameter was the general topics of the supportive essay assignments. As already seen these varied from the first to the second module, depending on the cases themselves. But more importantly, from a design perspective, this was a way of studying what may be required for an anchoring of the meta-issue reflections in the in-issues of the module to be present. For instance, in the supportive essay assignments of the first module the students would not be able to point out techniques used by Hamming in his development of the codes, or how these techniques contributed to this development, had they not been taught the related mathematics. In the second module the focus on anchoring in the supportive essay assignments was not as strong. The focus on the in-issues in the material was still very dominant, but in the essay assignment it was 'loosened' a little, the idea being to see to what extent an anchoring would still appear in the student essays.

Two modules instead of just one also opens up the possibility of making *predictions* from one module to the next, e.g. about the way in which students reflect on meta-issues – something which could only have been done in the sense of 'thought experiments' before the first module. Also two modules may show some degree of *replicability*, that is experiences as well as elements of design and methodology being transferable from one module to another – not from one population to another; at least in this research. Besides the above, another reason for having two modules with the same population was to see if, and if so then to what extent or in what way, such teaching modules on the history of mathematics may give rise to changes in students' beliefs about, images of, and attitudes towards mathematics. Since these are not likely to change from one day to the next, some substantial calendar time was needed to 'measure' possible changes. As mentioned, the students were given a questionnaire before the first module, a second in between the modules, and a third after the second module. The second and third questionnaires were closely related to the historical cases of the teaching modules – many of the questions being test questions of the students' understanding of both the meta-issues and in-issues of the modules – whereas the questions of the first questionnaire were of a more general nature, as illustrated in subsection 5.2. In order to measure any changes in the students' beliefs and images (and possibly also attitudes) the students were given a fourth questionnaire largely identical to the first. This was given to the students four months after the completion of the second module and the third questionnaire, which was one year after the first questionnaire.

5.5. *Comparison and analysis of the questionnaire answers*

The idea for comparing the students' questionnaires was to, first, create an overall picture of the students' answers to the first and fourth questionnaire, and while doing that look for patterns of changes. A few of the questions in the first and fourth questionnaire were present in the second and third questionnaires as well, though in more case-specific versions. More precisely, the questions of discovery versus invention were repeated in the following way:

- Are Hamming codes and Golay codes discovered or invented?
- Is RSA discovered or invented? Why?

And the question concerning whether mathematics is a science (scientific discipline) was repeated in the same phrasing through all four questionnaires. For these questions it was thus possible to follow the students' beliefs, and possible changes in these, more closely.

After having provided an overall picture of the students' beliefs and possible changes in these, the idea was to follow specific students more closely. The interviewees were of course the more interesting ones in this sense since their beliefs had been elaborated upon during the four rounds of interviews. And especially the focus group students were interesting since they also had been monitored more closely during the implementation of the modules.

6. EXAMPLES OF DATA AND RESULTS FROM THE EMPIRICAL STUDIES

Providing a long a thorough presentation, analysis, and discussion of the data from the two implementations of the teaching modules is not possible in a paper like this. For that reason, I shall instead illustrate the kind of answers the students gave to some of the essay assignments as well as to the questionnaires, and use these as a basis to present some of the results of the empirical studies. In doing this, I shall provide answers to the research questions posted in subsection 2.1, and the question of changes in beliefs as addressed in section 5. For further display and discussion of these results see Jankvist (2008a, 2008f, 2009c)[9]. All quotes from essays, questionnaires, and interviews have been translated from Danish.

6.1. *Data examples from students' essays in module 1*

In the following I shall focus on the students' answers to the second supportive essay assignment on objects and techniques in the early history of error correcting codes (see subsection 4.2). The answers to this assignment may be divided into three kinds. In the first kind no distinction was made between techniques already available to Hamming from the very beginning, and techniques which Hamming had to create himself in the process of developing and describing his codes. One group of students answered:

Group 6_1: The techniques Hamming uses to study codes: Hamming-distance, decoding with nearest neighbor, weight of the words, t-[error] detecting codes, t-error correcting codes, and the syndrome.

That is to say, a long list of concepts, methods, etc. related to the theory of error correcting codes. The second kind of answers was those taking into consideration only the already developed and available techniques, which was the original intent with the assignment. As an example of this, one group gave the following clear-cut answer:

Group 5_1: Hamming uses generalized concept of distance; elements of linear algebra; geometrical models; and unity n-dimensional squares.

The third kind of answers consisted of a mix of the two first, which is already available techniques and techniques created in the process of making. The majority of the answers to this essay assignment were of the third kind; only one group gave a clear-cut answer of the second kind (group 5_1

[9] The final description of the results will be available as part of my Ph.D. dissertation, which is expected ready during the summer of 2009, and which later shall be available at http://milne.ruc.dk/ImfufaTekster/.

above). Even though the example of answers of the first kind shown above does not concern the already established techniques, it does, in fact, still concern techniques. For instance, the group mentions Hamming-distance which is a metric and did serve as a technique for Hamming in order for him to describe the error correcting capabilities of his codes. In terms of redesign though, should the module be implemented again, it might be an idea to distinguish clearer in the phrasing of the assignment between already available techniques and techniques made in the process of development.

More interesting, however, is the fact that all groups to some extent mentioned either metric or Hamming-distance as a technique used by Hamming. This indicates some kind of anchoring of the meta-issue discussions concerning the birth and early development of error correcting codes in the acquired mathematical content (in-issues) of the module. Had the students not been taught the mathematics along with the history of error correcting codes, they would not have been able to discuss the history of the codes in terms of objects and techniques. Not in the way they did, anyway, by actually identifying used (old or new) techniques. From a design perspective this also suggests a way of 'ensuring' anchoring of the meta-issue discussions and reflections in the related in-issues.

6.2. *Data examples from students' essays in module 2*

Also for the second module I shall focus on the second supportive essay assignment, this one being on inner and outer driving forces in the history of public-key cryptography and the related number theory (see subsection 4.2). The typical answer to the first and second part of this essay was a list of mathematicians describing their personal motivation in relation to inner and outer driving forces. These lists would, however, vary substantially in both length and 'depth'. Let us take 'Diffie' as an example:

Group 3_2: Diffie studied mathematics in the USA, after which he worked different jobs related to computer technology. He was very fascinated by the Internet, which must have been an inner driving force for him. The outer driving force was that he could see that it would turn into a world wide computer network. He thought about the problems with trading via such a net, e.g. a person who wanted to buy things with a credit card might get into security problems with hacking etc.

Group 4_2: Diffie: He worked for a security agency where he made the key systems.

Now, despite being incorrect the latter quote is also quite short compared to the first. Fortunately, only a couple of the groups wrote answers like the latter, most groups gave longer descriptions and discussions along the line of the first. Of course, it was not always possible to give 'in-depth' descriptions of the motivations for the mathematicians, especially the older ones like Sun Zi and Euclid. However, some of the students would still perform somewhat reflected 'speculations':

Group 6_2: Euclid: mathematician, lived 300 B.C., wrote books about geometry, and three books about number theory. Euclid's algorithm: if you cannot find a gcd [greatest common divisor] which divides 2 numbers, Euclid's algorithm produces a solution.

As a function of time we may conclude that he basically worked out of an inner driving force since applied number theory didn't really exist at that time the way it does today. However, you may talk about knowledge as being power in this time [of history], and if he was striving for more power this may be considered an outer driving force.

Concerning the third question, I shall show only one quotation to illustrate the students' answers of this assignment. The quote is quite long, but to a large extent encompasses the more fragmented, and shorter, answers of some of the other groups:

Group 2_2: From the two previous questions, it is quite clear that the period under which the mathematicians in question lived had great influence on the mathematical research being conducted. In older times mathematicians did mathematics out of a desire of wanting to do so, often mathematics was a free time activity or even a hobby. The mathematical research was driven from within, the mathematics sought to solve problems for the sake of mathematical research itself. In comparison to the mathematicians of older date the mathematicians of newer date were influenced by outer driving forces. As Hardy also points out, war is an outer driving force for mathematics. Often wars have raised new questions which afterwards have been solved by scientists in terms of developing new areas within their respective fields, including mathematics. A[nother] clear outer driving force in this sense is money. This [war and money being outer driving forces] has, more or less, been the case for mathematicians like Diffie, Hellman, Ellis, Cocks, Rivest, Shamir, Adleman, etc. while the situation has been different for Euclid, Fermat, Euler, Gauss, Riemann, and Hardy.

The above quotes illustrate that students are (again), at some level, capable of carrying out discussions and reflections of the meta-issues related to the evolution and development of both pure and applied mathematics. One thing, however, which the students seemed to have problems with in the above assignment, was the differentiation between the inner and outer driving forces for mathematics as such, and the relations of these to the personal motivations of the single mathematician. For example, instead of saying what was an outer driving force for Diffie as a person (as e.g. group 3_2 did above), you can speak of the outer driving forces of cryptography as a whole, and then the personal ones for Diffie: e.g. there was an *outer* societal need for secure communication, on the *inner* lines the cryptographers were driven by the unsolved key-distribution problem, and Diffie *personally* had realized some of the possibilities of the then beginning Internet (the so-called ARPAnet). In the third round of interviews, after questionnaire three, I tried to have the students do such a distinguishing, but for many of them this seemed to be difficult, and some of them even seemed to find the 'split' somewhat artificial.

Concerning anchoring of the meta-issues in the related in-issues, the above quote on Euclid (group 6_2) shows the potential of such anchoring being present when the students talk about the Euclidian algorithm, and mention greatest common divisor (gcd) of two numbers. Actually what this quote illustrates nicely is that the students themselves seemed to feel a desire to bring in elements of mathematical in-issues in their answering of the essay assignments, and this even when there was no call for it built into the questions of the assignments (as was the case with the previous assignment on objects and techniques). This was something which also occurred frequently in the main essay assignment, when the students were to give the who-and-when and the why-and-how accounts for the historical case.

Thus, even though the constraints on the in-issue anchoring was loosened in form of choice of 'general topics', the results of the first teaching module were still reproduced, and the empirical investigation as a whole seems to suggest some degree of replicability. I shall address this in section 7, but for now let us turn to the question of students' beliefs.

6.3. *Data examples of students' changes in beliefs*

I shall exemplify the students' changes in beliefs by means of the somewhat philosophical question on mathematics as being discovered or invented. Due to the limitation of space, I shall focus on the overall picture of the students' beliefs (for an in-depth discussion of individual students' changes in beliefs, the reader is referred to my dissertation as mentioned in footnote 9). Due to the readability of the questionnaire results and the relatively small population of the class, the students' answers have been indexed in the following manner:

$$\text{one} < \text{few} < \text{some} < \text{many} < \text{the majority} < \text{the } \textit{vast} \text{ majority},$$

a partition which in percentage intervals roughly corresponds to $0-5\%$, $6-15\%$, $16-35\%$, $36-50\%$, $51-80\%$, and $81-100\%$, respectively.

In the first questionnaire, the majority of the students believed that mathematics in general is something you discover. Examples are: "Discover. I don't think you can invent mathematics – it is something 'abstract' you find with already existing things."; "Good question... very philosophical. I think there are many different standpoints to this. I personally believe that it is something you discover. Numbers and all the discoveries already made are all connected. So for me it is more a world you enter into than one you make." Only a few students believed that it is something you invent. Some students, though, believed it might be a mix of the two, e.g. the one saying: "Many things might begin as an invention, but afterwards they are explored and people discover new elements in the 'invention' in question". An interesting aspect of this question is the relation of the students' answers to the previous question on the negative numbers as something being discovered or invented (see subsection 5.2). Concerning this, many of those who believed that negative numbers were something discovered stuck to this point of view for mathematics in general. Approximately as many, however, of those who believed that negative numbers were discovered, and a few of those who believed them to be invented, said "both" for mathematics in general. And some of those who believed negative numbers to be invented believed mathematics in general to be discovered.

Concerning the second questionnaire and concerning the students' answers to the related question of discovery and invention of error correcting codes, the class at this time was about equally split between students who believed in discovery and students who believed in invention. A few examples are: "Discovered because the codes have been there all the time, you just haven't known them before"; "Discovered. The codes were there, and it was discovered that there was a certain connection, and then they were called something"; "Invented, I think, because it sounds awkward to discover a code"; "Invented since you, for example, can't go find a binary number system [in nature]"; "Invented – if not they shouldn't be allowed to patent them". A few students answered "both", one of whom argued: "Both, I'd say. They discovered the codes because they used the arithmetical rules and stuff they had invented". A few students did not answer the question.

In the third questionnaire, however, the majority of the students believed RSA to be invented. Some said that it was invented due to a very specific need of cryptography and safety in society; due to the need of solving the key-distribution problem; or due to the necessity of dealing with situations as those arising in war, for instance. Other examples of arguments for invention are: “Invented. You’ve put some theorems together and called them RSA”; “Invented I think, because it is a very specific way of coding and decoding”. Some students said that RSA is discovered. The arguments given were either general or case-specific: “Discovered. All mathematics has always existed”; “It is discovered since I don’t believe that numbers can be invented: all numbers exist in all connections, they just haven’t been found yet”; “RSA is discovered within public-key cryptography since public-key cryptography already was made then”; “Discovered. You knew very well already that there was a one-way function to discover (public-key cryptography)”. A few students said that it is a mix of discovery and invention. And a few students could not make up their minds: “I can’t seem to agree with myself upon this”; “Well, here I might say invented... or... no... I’m not sure. You get more and more confused the more you think about it”. One student did not answer the question.

In the fourth questionnaire, many students actually answered “both”, or that it is a combination. Examples of answers are: “Both actually, but at first you discover and from that you invent things”; “Both. You have a problem to solve → invent. Solve the problem mathematically → discover”; “It is a mix, and I don’t think you can give a definite answer to it” One of the answers relates directly to the teaching modules: “It is both. You discover objects but invent techniques. So the question of discovery or invention depends on the scale of the perspective upon which you look at it” The rest of the class was equally divided between believing invention and believing discovery. However, an analysis of these answers show that even those students who answered “discovery” still seemed to think that it is some kind of a combination. Furthermore, a general tendency seems to be that students believe discovery to precede invention. The students who said invention did generally not provide arguments for their belief. However, all these students’ beliefs, except one, were consistent with their answers in the previous question on negative numbers being discovered or invented, where some of them already had provided their reasons. Some students who answered “invented” in regard to the negative numbers believed it to be a mix for mathematics in general, and so did a few students who favored discovery of the negatives. A few students who believed in invention of the negatives now answered “discover”. Only one student who was in favor of discovery of the negatives believed mathematics in general to be invented. And one student answered “both” in both questions.

The above display of students’ answers does indicate some form of development in their beliefs about the question of discovery versus invention of mathematics. In the first questionnaire students seem to favor discovery, in the second questionnaire the students are about equally split between discovery and invention of error correcting codes, in the third questionnaire the majority believes invention in the case of RSA, and in the fourth questionnaire the class is split three ways, one third now believing it to be a mix. Now, the important thing here is not whether the students believe in discovery or in invention. The important thing is whether their beliefs are reflected, and if they have given thought to them before answering – it is in this sense that the presence of development in beliefs is interesting. The fact that one third believe it to be a mix in the fourth questionnaire show that some reflection has taken place, and the two last answers presented for questionnaire three above, where the students are in doubt and cannot make up their mind, also show a larger degree of reflection being presents in the students’ views. Furthermore, the development of students’ beliefs on negatives and mathematics in general from the first to the fourth questionnaire indicates an increase in the level of consistency in the students’ beliefs. Another change appears to be that more students in the later questionnaires seem to feel a greater need to justify their beliefs, some even by referring to specific elements from the modules, as when one student refers to objects and techniques in questionnaire four (see above). Some of these phenomena also appeared in the follow-up interviews, and may thus be confirmed by performing triangulations between the different questionnaires and interviews.

7. Discussion and recapitulation

One important issue which needs to be touched upon as part of this discussion is what questions that could *not* have been answered without carrying out the above described piece of empirical research. First of all, before conducting the research there was no evidence of upper secondary students being capable of performing meta-issue reflections of mathematics and its evolution at all. In this respect, the research studies and the results described above provide an ‘existential proof’ of meta-issue reflections being possible as well as it being possible to anchor such reflections in in-issues. The next step then is to make this existential proof into a more constructive one. By this I mean to identify what made these reflections possible, for example how much of the 'success' can be ascribed to the teaching material and the set up, and how much to the teacher and her way of teaching the module. If we are to deal with the critique by Gulikers and Blom as presented in section 1 such matters need to be clarified in order to transfer the experiences to "other teachers, classes, and types of schools". It is not just a matter of describing what works, but also about describing why what works actually works (Lester, Jr., 2005). For the remaining part of this section, I shall address these matters.

The results of the two empirical studies suggest that the students to some extent were capable of doing meta-issue reflections and carrying out meta-issue discussions in their groups. Concerning the terms on which the students were able to do this, their work with the final essay assignments seems to have played an important role. Besides providing evidence of meta-issue reflections, the students' hand-ins of these assignments also evidenced an anchoring of these meta-issue reflections in the in-issues of the modules. In the first module, the discussion of objects and techniques, as assumed, to a certain degree ensured this. But also in the second module, where the 'ensuring' of this had been loosened, the anchoring was still present in the hand-in essay assignments. Thus, not only the essay assignments played an important role, also the strong focus on the in-issues in the teaching material seems to have played an important part in this replicability of results. Of course, the teacher herself and her teaching were also factors not to be neglected. However, the interviews with the teacher showed that she was not too interested in the history of mathematics, and actually she focused more on the meta-issues concerning application in her teaching (the video tapings document this). So the connections between the meta-issues concerning history and the in-issues were mostly done by the students themselves based on the teaching material. In this respect, the setting of having the students work on the essay assignments in groups seemed to be a key element of the success in terms of history as a goal. The video of the focus group shows that the students, before writing down their final answer to the assignments, would have lengthy discussions in which the focus shifted between historical, sociological, philosophical as well as mathematical discourses, and in which they often would refer to the teaching material (and sometimes the Internet) when in doubt.

Another matter concerns the choice of historical cases. Even though there is no unique solution to finding and choosing a case for a teaching module concerned with the illustration of history as a goal, some criteria may still be listed. Firstly, the mathematical content of the cases, i.e. the in-issues, should be explainable to the students. It is extremely important that the mathematics does not become a hurdle for the students in their engaging in reflections on the history. However, the history should not be a hurdle in the learning of the mathematics either, as pointed to by van Amerom (2002, p. 297) and Bakker (2004, p. 266). Some of the students' problems with the described supportive essay assignment of the second teaching module above seem to suggest something similar: not only the in-issues should be adjusted according to the educational level; the meta-issues should be so as well. Or put another way, the elements of mathematics and the elements of history as well as their level of difficulty should be balanced against each other. Secondly, the cases should be exemplary in illustrating meta-issues of the evolution and development of mathematics which concern the history of mathematics in general (the idea of 'general topics'). Looking at the history of mathematics through epistemic objects and techniques as done in the first teaching module is a working tool which applies to the history of mathematics as a whole. The same goes for the meta-issue reflections on inner and outer driving forces in the second teaching module, although it may be more visible in cases concerning applied mathematics. So even though the students were working with specific cases and specific in-issues from the history of mathematics, they were, in fact, working with meta-issues which concern the evolution and development of mathematics as a whole. Thirdly, the cases should be of such a nature that it is possible, and preferably even natural, to have the meta-issue reflections anchored in the in-issues of the case. Also, this was a quality which both cases of the teaching modules possessed, and which was verified by looking at the students' answers to the final essay assignments and the videos of the focus group. (Of course, this list is not meant to be an exhaustive one, other criteria may thought of and listed.)

Despite the present study on students' beliefs not being a large scale study, as mentioned the population was 23 students, it still seems possible to draw some more or less general conclusions. As illustrated in terms of the discovery versus invention question, one change in the students' beliefs seems to be that the different beliefs appear to be more consistent in the fourth questionnaire, and another change appears to be that the students seem to feel a need to justify their beliefs to a greater extent after the modules. These phenomena do not only occur in the question addressed above, they also occur in some of the other questions mentioned in subsection 5.2. An example of this is the following answer by a student to the question of whether mathematics can become obsolete: “Not all of mathematics, but parts of it which you might not need anymore. Caesar encryption has for instance become obsolete today, because it is too simple. You have a need to invent a new cryptographic method.” Besides justifying his belief, this student also does another thing, namely to exemplify his view by means of the treated historical cases – and this in the fourth questionnaire which did not directly relate to either of the modules. The students’ answers to the questionnaires as well as the interviews contain several examples of such links to the historical cases of the modules, thus providing a means for attributing certain changes to these.

According to Lester, Jr. (2002, p. 352), Kath Hart at a PME conference once asked: “Do I know what I believe? Do I believe what I know?” Lester’s version of this question is: “Do students know what they believe?” And his answer is: “I do not think most students really think much about what they believe about mathematics and as a result are not very aware of their beliefs” (Lester, Jr., 2002, p. 353). Certainly this might have been the case when the students answered the first questionnaire, but when the students reached the fourth questionnaire the situation appears different. For example, the question of discovery versus invention was not an explicit part of the teaching materials and implementations of the modules; it was only present in the questionnaires and follow-up interviews. Nevertheless, the modules provided a setting in which the students were given the opportunity to reflect upon their views and beliefs concerning this question, as well as others, and the modules also provided specific historical cases for the students to ‘test’ these views and beliefs on. As mentioned earlier, having students reflect upon their beliefs is indeed a ‘goal’ in itself. However, reflection and the ability to *perform* reflection are also considered to be major factors in the changing of beliefs in general (Cooney, Shealy, and Arvold., 1998; Cooney, 1999). Thus, if students are to have their beliefs ‘molded’, ‘shaped’, or changed in such a fashion that they fit better the more ‘goal’ oriented descriptions of the KOM-report, then a setting of a suitable scene for enabling them to perform reflections appears to be a necessity.

A few comments on the methodology being restricted to history as a goal are also in order. Firstly, had the focus of the empirical studies been history as a tool instead of history as a goal, and then the above ‘list’ of design criteria would have looked different. For instance, if the focus had been on teaching a specific mathematical concept through its history, then the question of whether or not this history was exemplary in the sense described above would have been less important since the case would already have been given by the concept in question. Secondly, any kind of research design and research methodology has its limitations, that is to say the design and methodology may only be applicable to answer some questions and not others. For example, some history as a tool questions may not be answered within the design and methodology of the research studies described above. An example of this are the history as a tool questions concerning the so-called historical parallelism between the evolution and the learning of mathematical concepts, theories, methods, etc. In order to investigate such questions the design and set up must be of a quite different nature, enabling the students to follow the main road of history in their learning process[10].

8. PERSPECTIVES

With this paper it is my hope to have contributed a little to the discussion of the role of empirical research in the 'whys' and 'hows' of using history in mathematics education, especially in terms of design and methodology relating to the use of history as a goal. Another obvious question to ask in relation to these matters is what the possible benefits of increasing the amount of empirical research in the field of using history in mathematics education might be. One answer to this question was provided to me by Abraham Arcavi, whom I interviewed in June 2007 in Iceland. When discussing the need for empirical research within the field of using history in mathematics education, often also just referred to as HPM for the *History and Pedagogy of Mathematics* group,

Arcavi said:

> HPM still needs much more empirical research on teaching and learning related to history than it is the case now, and there is no lack of research questions to pursue. [...] research, as I envision it, would provide insights which confirm, extend or challenge some of our assumptions and proposals, it may reveal directions not yet pursued and it would certainly sharpen our own views and future plans. (Arcavi in Jankvist, 2008d, pp. 17-18)

Concerning the empirical studies of this paper a few suggestions may be made in terms of possible benefits, both relating to the use of history as a goal

[10] For examples of studies touching upon this see, for instance, Harper, 1987; Sfard, 1995; and Tzanakis and Kourkoulos, 2007.

and the use of history as a tool. Some studies have been made concerning teachers' beliefs about and images of the evolution and development of mathematics (e.g. in Furinghetti, 2007; Goodwin, 2007; Philippou and Christou, 1998). However, studies on students' beliefs about these matters appear scarce, at least in the English literature. The empirical investigation presented in this paper to some extent suggests what may be gained from increasing the body of research on students' beliefs on such matters. For one, we might be able to answer questions about how to provide students with more sound images of mathematics, both as a historically rooted discipline and as a discipline which is still evolving today. Such matters are sometimes also discussed in terms of providing students with 'mathematical appreciation' or 'mathematical awareness' (e.g. Furinghetti, 1993; Niss, 1994; Ernest, 1998). Another question related to the empirical studies concerns whether it might be possible to say something about changes in beliefs as a function of the students' capabilities in performing meta-issue discussions and reflections, as well as the dependence of students' beliefs on students' actual understanding of related in-issues. The presented empirical study of this paper might provide some suggestions to answers of these questions, that is if the data be reinterpreted from this point of view, but certainly one study alone is not enough to answer such general questions.

Also in terms of history as a tool something may be said. One type of history as a tool arguments are the motivational or affective arguments (for other types, see Jankvist, 2009a). The fact that the historical cases of the two teaching modules dealt with applications of mathematics, and furthermore applications which the students could recognize from their own everyday life, seemed to have an effect on their motivation to engage in the meta-issue reflections (for a discussion of this, see Jankvist, 2008b and Jankvist, 2009b). As mentioned earlier, also the fact that the history of cryptography links mathematics and war close together was something which seemed to motivate some students in their participation in the modules. Thus, even though the purpose of the history in the teaching module was concerned with history as a goal, the choice of cases seemed to have a side effect in terms of history as a tool.

Of course, looking also at some of the previously provided lists of 'old' arguments for (and approaches to) using history[11] through empirical research lenses, and putting these arguments to the (empirical) test, may reveal new insights in terms of which roads to travel in the future by confirming

[11] For example, those in Fauvel (1991) and in Tzanakis and Arcavi (2000).

these arguments, or by revealing which ones that turn out to be blind alleys; something which mere speculations and armchair research cannot do. Staying in the road system metaphor, such studies may also show us unknown intersections between the already known paths. For instance, in terms of the piece of empirical research described in this paper it would be interesting to see if the students who understand the mathematical in-issues the better are also able to better reflect on the meta-issues than those students who do not grasp the mathematics of the modules. Should it turn out to be so, this may reveal a new intersection between the path of history as a tool and the one of history as a goal. And not only the 'old' and often 'propagandistic', to use the words of Siu and Tzanakis (2004), arguments of the 'whys' and 'hows' may be tested in such manner. Also all the suggestions from the body of literature on what mathematical contents to involve, which original sources to read, and so on, may be tested in order to "confirm, extend or challenge" the assumptions and claims made about the benefits of using exactly these. Or to phrase it differently, thus ending where I began this paper, the next step will be to test – empirically – the 'effectiveness' of the proposed uses of the history of mathematics in mathematics education.

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REFERENCES

Bakker, A. (2004). *Design research in statistics education – On symbolizing and computer tools*. Ph.D. thesis, The Freudenthal Institute, Utrecht.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer Academic Publishers. Edited and translated by Nicolas Balacheff, Martin Cooper, Rosamund Sutherland, and Virgina Warfield.

Cooney, T. J., Shealy, B. E. & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education 29*(3), 306–333.

Cooney, T. J. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics 38* (1-3), 163–187.

Diffie, W. & Hellman, M. E. (1976). New directions in cryptography. *IEEE Transactions on Information Theory 21,* 644–654.

Epple, M. (2000). Genisis, Ideen, Institutionen, mathmatische Werkstätten: Formen der Mathematikgeschichte – Ein metahistorischer Essay. *Mathematische Semesterberichte 47,* 131–163.

Ernest, P. (1998). Why teach mathematics? – The justification problem in mathematics education. In: J. H. Jensen, M. Niss, and T. Wedege (Eds.), *Justification and Enrollment Problems in Education Involving Mathematics or Physics* (pp. 33–55). Roskilde, Denmark: Rosklide University Press,.

Fauvel, J. & van Maanen, J. (Eds.). (2000). *History in Mathematics Education – The ICMI Study*. Dordrecht: Kluwer Academic Publishers.

Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics 11*(2), 3–6. Special Issue on History in Mathematics Education.

Fréchet, M. (1906). Sur quelques points du calcul fonctionnel. *Rend. Circolo Mat. Palermo 74*: 1–74.

Freudenthal, H. (1991). *Revisiting Mathematics Education – China Lectures*. Dordrecht: Kluwer Academic Publishers.

Furinghetti, F. (1993). Images of mathematics outside the community of mathematicians: evidence and explanations. *For the Learning of Mathematics 13*(2), 33–38.

Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics 66*(2), 131–143. Special issue on the history of mathematics in mathematics education: Theory and practice.

Golay, M. J. E. (1949). Notes on digital coding. *Proceedings of the IRE 37*, p. 657.

Goodwin, D. M. (2007). *Exploring the relationship between high school teachers mathematics history knowledge and their images of mathematics*'. Ph.D. thesis, University of Massachusetts, Lowell.

Grassmann, H. (1844). *Die lineale Ausdehnungslehre*, Leipzig: Otto Wigand, 1844.

Gravemeijer, K. & Cobb, P. (2006). Outline of a method for design research in mathematics education. In: J. van den Akker, K. Gravemeijer, S. McKenney, and N. Nieveen (Eds.): *Educational Design Research* (pp. 17–51). Abingdon, Oxon: Routledge.

Gulikers, I. & Blom, K. (2001). 'A historical angle', a survey of recent literature on the use and value of the history in geometrical education. *Educational Studies in Mathematics 47*(2), 223–258.

Hamming, R. W. (1950). Error detecting and error correcting codes. *Bell System Technical Journal 29*, 147–160.

Harper, E. (1987). Ghost of Diophanthus. *Educational Studies in Mathematics 18* (1), 75–90.

Jankvist, U. T. (2007). Empirical research in the field of using history in mathematics education: Review of empirical studies in HPM2004 & ESU4'. *Nomad 12*(3), 83–105.

Jankvist, U. T. (2008a). Evaluating a teaching module on the early history of error correcting codes. In: M. Kourkoulos and C. Tzanakis (Eds.), *Proceedings 5th International Colloquium on the Didactics of Mathematics*. Rethymnon: The University of Crete. In press.

Jankvist, U. T. (2008b). History of modern mathematics and/or modern applications of mathematics in mathematics education. In: *Proceedings HPM2008 (CD-ROM)*. The HPM Group.

Jankvist, U. T. (2008c). *Kodningsteoriens tidlige historie – et undervisningsforløb til gymnasiet*, No. 459 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: The Early History of Error Correcting Codes – a Teaching Module for Upper Secondary School. 73 pages.

Jankvist, U. T. (2008d). Proceedings HPM2004&ESU4: empirical research on using history of mathematics in mathematics education. *HPM Newsletter* (67), 15–18.

Jankvist, U. T. (2008e). *RSA og den heri anvendte matematiks historie – et undervisningsforløb til gymnasiet*, No. 460 in Tekster fra IMFUFA. Roskilde: IMFUFA. English translation of title: RSA and the History of the Applied Mathematics in the Algorithm – a Teaching Module for Upper Secondary School. 116 pages.

Jankvist, U. T. (2008f). A teaching module on the history of public-key cryptography and RSA. *BSHM Bulletin 23*(3), 157–168.

Jankvist, U. T. (2009a). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics*. In press (available online first).

Jankvist, U. T. (2009b). History of modern applied mathematics in mathematics education. *For the Learning of Mathematics 29*(1). In press.

Jankvist, U. T. (2009c). Students' beliefs about the evolution and development of mathematics. *Proceedings from the CERME6* Working Group 15. Preprint.

Kjeldsen, T. H. (2008). Egg-forms and measure-bodies: different mathematical practices in the early history of the modern theory of convexity. *Science in Context 22*(1), 85-113.

Lester, Jr., F. K. (2002). Implications of research on students' beliefs for classroom practice. In: G. C. Leder, E. Pehkonen, and G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?,* Chapter 20 (pp. 345–353). Dordrecht: Kluwer Academic Publishers.

Lester, Jr., F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM 37*(6), 457–467.

Niss, M. & Jensen, T. H. (eds.). (2002). *Kompetencer og matematiklæring – Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Undervisningsministeriet. Uddannelsesstyrelsens temahæfteserie nr. 18. English translation of title: Competencies and Learning of Mathematics – Ideas and Inspiration for the Development of Mathematics Education in Denmark.

Niss, M. (1980). Nogle aspekter for matematikundervisningen i de gymnasiale uddannelser frem til 1990. *Normat 28*(2), 52–60, 87. English translation of title: Some aspects for mathematics education in upper secondary school approaching 1990.

Niss, M. (1994). Mathematics in society. In: R. Biehler, R. W. Scholz, R. Sträßer, and B. Winkelmann (eds.): *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Kluwer Academic Publishers, pp. 367–378.

Niss, M. (2001). Indledning. In: M. Niss (Ed.), *Matematikken og Verden*, Fremads debatbøger – Videnskab til debat. København: Forfatterne og Forlaget A/S. English translation of title: Introduction to the book: Mathematics and the World.

Philippou, G. N. & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics 35*(2), 189–206.

Rheinberger, H.-J. (1997). *Toward a History of Epistemic Things: Synthesizing Proteins in the Test Tube*. Stanford: Stanford University Press.

Rivest, R. L., Shamir, A. & Adleman, L. (1978). A method for obtaining digital signatures and public-key cryptosystems. *Communications of the ACM 21*, 120–126.

Rosen, K. H. (2003). *Discrete Mathematics and Its Applications*. New York: McGraw Hill, fifth edition.

Sfard, A. (1995). The development of algebra: confronting historical and psychological perspectives. *Journal of Mathematical Behaviour 14*, 15–39.

Shannon, C. E. (1948). A mathematical theory of communication I, II. In: D. Slepian (Ed.), *Key Papers in The Development of Information Theory*, *Vol. 27*. New York: IEEE Press, pp. 5-18, 19-29. (Pages 379–423 and 623–656 in the original publications by Shannon.)

Singh, S. (1999). *The Code Book – The Secret History of Codes and Codebraking*. London: Fourth Estate.

Siu, M.-K. & Tzanakis, C. (2004). History of mathematics in classroom teaching – appetizer? main course? or dessert?. *Mediterranean Journal for Research in Mathematics Education 3*(1-2), v–x. Special double issue on the role of the history of mathematics in mathematics education (proceedings from TSG 17 at ICME 10).

Thompson, T. M. (1983). *From Error-Correcting Codes through Sphere Packings to Simple Groups*, No. 21 in The Carus Mathematical Monographs. The Mathematical Association of America.

Toeplitz, O. (1927). Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen. *Jahresbericht der deutschen Mathematiker-Vereinigung* XXXVI, 88–100.

Tzanakis, C. & Arcavi, A. (2000). Integrating history of mathematics in the classroom: an analytic survey. In: J. Fauvel and J. van Maanen (Eds.), *History in Mathematics Education,* Chapter 7 (pp. 201–240.). The ICMI Study. Dordrecht: Kluwer Academic Publishers.

Tzanakis, C. & Kourkoulos, M. (2007). May history and physics provide a useful aid for introducing basic statistical concepts?. In: F. Furinghetti, S. Kaijser, and C. Tzanakis (Eds.), *Proceedings HPM2004 & ESU4* (pp. 284–295). Uppsala Universitet, revised edition.

Undervisningsministeriet (2007). Vejledning: Matematik A, Matematik B, Matematik C. Bilag 35, 36, 37. Guidance for upper secondary mathematics teachers by the Danish ministry of education.

van Amerom, B. A. (2002). *Reinvention of early algebra – Developmental research on the transition from arithmetic to algebra*. Ph.D. thesis, The Freudenthal Inistitute, Utrecht.

Wells, D. (2005). *Prime Numbers – The Most Mysterious Figures in Math*. Hoboken, New Jersey: John Wiley & Sons, Inc.

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