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EXPLORING LEARNING OPPORTUNITIES FOR PRIMARY TEACHERS: THE CASE OF KNOWLEDGE FOR TEACHING EARLY ALGEBRA

RESUMEN

Comprender cómo constituir y desarrollar oportunidades para que los maestros de primaria enseñen álgebra temprana a niños sigue siendo un importante vacío en la investigación. En este artículo presentamos los resultados de un programa de investigación desarrollado en Brasil durante los últimos cinco años. Nuestro objetivo es discutir cómo surgieron oportunidades de aprendizaje profesional cuando los maestros planificaron, discutieron y analizaron colectivamente lecciones que involucraban diferentes significados del símbolo de igualdad y el desarrollo del pensamiento funcional. Desarrollados desde la perspectiva de una investigación cualitativa-interpretativa, los datos analizados consisten en documentos curriculares, protocolos para la resolución de tareas formativas, audios y videos recopilados durante procesos de formación docente para docentes en servicio. Los resultados destacan que las tareas de aprendizaje profesional, combinadas con las acciones de los formadores de docentes durante las discusiones colectivas, favorecieron a los profesores en servicio para diferenciar y comprender el razonamiento de los estudiantes. Se discuten algunas implicaciones para la formación docente, así como el desarrollo profesional de los maestros de primaria, especialmente en relación con el pensamiento algebraico temprano, ya que los maestros normalmente no tienen la oportunidad de estudiar estos contenidos en sus propias experiencias en la escuela.

PALABRAS CLAVE:

- *Oportunidad de aprendizaje docente*
- *Formador de docentes*
- *Tareas de aprendizaje profesional*
- *Práctica docente de primaria*
- *Álgebra temprana*

ABSTRACT

Understanding how to constitute and develop opportunities for primary teachers teach early algebra to younger children is still an important research gap. In this paper, we bring results of a research program developed in Brazil over the past five years. We aim to discuss how professional learning opportunities emerged when teachers collectively planned, discussed, and analyzed lessons involving different meanings of the equality symbol and the development of functional thinking. Developed from the perspective of a qualitative-interpretative research, data analyzed consists of curriculum documents, protocols

KEY WORDS:

- *Teacher learning opportunity*
- *Teacher educator*
- *Professional learning tasks*
- *Primary teacher practice*
- *Early algebra*

Alessandro Jacques Ribeiro, A. J., Aguiar, M., Trevisan, A. L. y Elias, H. R. (2023). Exploring learning opportunities for primary teachers: the case of knowledge for teaching early algebra. *Revista Latinoamericana de Investigación en Matemática Educativa 26*(3), 327-356. https://doi.org/10.12802/relime.23.2633. Recepción: Junio 6, 2022 / Aceptación: Septiembre 6, 2024.

for the resolution of formative tasks, audios and videos collected during teacher education processes for in-service teachers. The results highlight that professional learning tasks, combined with the actions of teacher educators during collective discussions, favored in-service teachers to differentiate and understand the students' reasoning. Some implications for teacher education as well as the professional development of primary teachers are discussed, especially regarding early algebra thinking, because teachers do not normally have the opportunity to study these contents on their own experiences in school.

RESUMO

Compreender como constituir e desenvolver oportunidades para professores primários ensinarem álgebra precoce a crianças ainda é uma importante lacuna de pesquisa. Neste artigo, trazemos resultados de um programa de pesquisa desenvolvido no Brasil nos últimos cinco anos. Nosso objetivo é discutir como as oportunidades de aprendizagem profissional surgiram quando os professores planejaram, discutiram e analisaram coletivamente aulas envolvendo diferentes significados do símbolo da igualdade e o desenvolvimento do pensamento funcional. Desenvolvidos na perspectiva de uma pesquisa qualitativo-interpretativa, os dados analisados consistem em documentos curriculares, protocolos para a resolução de tarefas formativas, áudios e vídeos coletados durante os processos de formação de professores em serviço. Os resultados destacam que as tarefas de aprendizagem profissional, combinadas com as ações dos formadores de professores durante as discussões coletivas, favoreceram os professores em serviço para diferenciar e compreender o raciocínio dos alunos. São discutidas algumas implicações para a formação de professores, bem como para o desenvolvimento profissional dos professores primários, especialmente no que diz respeito ao pensamento inicial da álgebra, pois os professores normalmente não têm a oportunidade de estudar esses conteúdos em suas próprias experiências na escola.

PALAVRAS CHAVE:
- *Oportunidade de aprendizagem de professores*
- *Formador de professores*
- *Tarefas de aprendizagem profissional*
- *Prática de professores primários*
- *Álgebra precoce*

RÉSUMÉ

Comprendre comment constituer et développer des opportunités pour les enseignants du primaire d'enseigner l'algèbre précoce aux jeunes enfants reste une lacune importante dans la recherche. Dans cet article, nous apportons les résultats d'un programme de recherche développé au Brésil au cours des cinq dernières années. Nous visons à discuter de la façon dont les opportunités d'apprentissage professionnel ont émergé lorsque les enseignants ont collectivement planifié, discuté et analysé des leçons impliquant différentes significations du symbole d'égalité et le

MOTS CLÉS:
- *Opportunité d'apprentissage des enseignants*
- *Formateur d'enseignants*
- *Tâches d'apprentissage professionnel*
- *Pratique des enseignants du primaire*
- *Algèbre précoce*

développement de la pensée fonctionnelle. Développées dans la perspective d'une recherche qualitative-interprétative, les données analysées consistent en des documents curriculaires, des protocoles pour la résolution de tâches formatives, des audios et des vidéos collectés au cours des processus de formation des enseignants pour les enseignants en service. Les résultats mettent en évidence que les tâches d'apprentissage professionnel, combinées aux actions des formateurs d'enseignants lors de discussions collectives, ont favorisé les enseignants en poste pour différencier et comprendre le raisonnement des élèves. Certaines implications pour la formation des enseignants ainsi que le développement professionnel des enseignants du primaire sont discutées, en particulier en ce qui concerne la réflexion précoce sur l'algèbre, car les enseignants n'ont normalement pas la possibilité d'étudier ces contenus sur leurs propres expériences à l'école.

1. Introduction

Research findings have showing us the necessity to invest in the continuing education of teachers in order to establish connections and interlocutions between their professional knowledge (Ball & Cohen, 1999) and the teaching practices of these teachers. This stems from an understanding that teachers continue to learn while exercising their professional practices (Webster-Wright, 2009) and that there are ongoing possibilities of building a body of mathematical knowledge for teaching early algebra (Ball et al., 2008; Pincheira & Alsina, 2021).

Understanding how the teacher learning process takes place and how it develops throughout their career (Webster-Wright, 2009) is strongly linked to the learning opportunities that teachers experience. The term "learning opportunity" has been researched for a long time regarding primary school students (Heyd-Metzuyanim et al., 2016). However, in teacher education, these studies are more recent, whether related to pre-service (Tatto & Senk, 2011) or in-service education (Ribeiro & Ponte, 2019).

In terms of mathematical knowledge, developing students' algebraic thinking at the early stages of schooling is fundamentally important in order to open doors to the study of algebra in subsequent years (Kieran et al., 2016). Other researchers reveal that there is knowledge that teachers need to mobilize and (re)structure to be able to explore this theme in their classrooms (Ponte & Branco, 2013). Thus, our study is centered on understanding what mathematical knowledge teachers need to have to support student learning (Ribeiro et al., 2021) regarding early algebra. The approval of the Common National Curriculum Base - BNCC (Brasil, 2017), a document that indicates what should be taught in Basic Education in Brazil, introduces Algebra as a new thematic unit to be worked on in Mathematics. This recent inclusion of Algebra in the curriculum of early years follows an international trend based on the realization that young students are already able to think algebraically, and that leading them to this type of reasoning, in addition to being relevant, is essential if what is desired is that students go beyond performing operations and solving problem situations (Ferreira, 2017).

Although the Early Algebra movement has encouraged investigations into the learning of algebra by early-year students (Warren et al., 2016), there are few studies that address the issue from the point of view of teachers' professional practice approaching algebraic thinking (Jacobs et al., 2007) or to professional knowledge for teaching Early Algebra (Pincheira & Alsina, 2021). Some challenges can make it difficult for early-year students to think algebraically, among them is the fact that teachers generally did not have access to the necessary knowledge in order to teach algebra to younger students, whether in their initial or continuing education (Blanton, 2008).

From our point of view, one of the prominent paths may be through classroom situations that contribute to a reflective context for teachers (Silver et al., 2007), and that consider (i) the content addressed in teacher education (Desimone, 2009), (ii) the importance given to the elaboration of mathematical tasks and their development in the classroom (Christiansen & Walther, 1986), (iii) the possibilities for teachers to plan lessons collectively (Serrazina, 2017) and apply them through an exploratory teaching approach (Canavarro et al., 2012) and, finally, (iv) to reflect on what happened as well as think of new practices for their classes (Ponte, 2005).

In our study, we emphasize an approach that explores the different meanings of the equality symbol (Kieran 1981; Trivilin & Ribeiro, 2015) and the study of functional thinking (Carpenter et al., 2005; Zapatera Llinares, 2018) starting in the early years of elementary school. Therefore, this article aims to *understand how learning opportunities are constituted and developed so that Primary Mathematics Teachers (PMTs) can approach early algebra in the first years of elementary school*. To this end, we seek to answer the following research questions: (RQ1) *What professional learning opportunities arise when teachers collectively plan and analyze classes involving different meanings of the equality symbol and the development of functional thinking?* (RQ2) *What mathematical knowledge do teachers mobilize and build to teach early algebra when they experience collective opportunities for professional learning?*

2. Theoretical Framework

2.1. *Professional Learning Opportunities for Teachers*

In order to understand how opportunities for teachers to learn are constituted, we first need to understand how teachers learn. For this, we adopted in our study an understanding that teacher learning is located in their daily practice, including not only classroom moments, but also those that are focused on planning, evaluating and collaborating with colleagues and others (Davis & Krajcik, 2005), and also understand that teacher learning is distributed among individuals and artifacts, as is the case of tasks developed for their education (Putnam & Borko, 2000).

Based on these principles, Ribeiro & Ponte (2020) organized the Professional Learning Opportunities for Teachers (PLOT) model, which constitutes a theoretical-methodological model with the purpose of (i) organizing the design of formative processes that aim to promote learning for teachers and (ii) generate opportunities for teachers to learn during these formative processes. The model is organized from three interconnected domains: (a) Role and Actions of Teacher Educator (RATE), (b) Professional Learning Tasks for Teachers (PLTT), and (c) Discursive Interactions Among Participants (DIAP); that collectively contribute to the creation of PLOTs, from certain contexts (Figure 1).

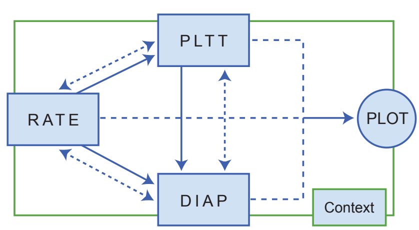


Figure 1. PLOT Model (Ribeiro & Ponte, 2020, p. 4).

When considering teacher learning situated and mediated by instruments, people and context, the PLOT model establishes that its domains are decisive in the design, implementation and evaluation of formative processes that aim to provide opportunities for teachers to learn from one another.

Regarding the *RATE domain* indicates the skills needed by teacher educators, such as selecting and using appropriate tools and resources for teaching (Zaslavsky, 2008), designing formative processes considering the characteristics of the local context (Desimone, 2009), considering mediation actions and conducting education in an exploratory teaching environment (Ponte, 2005), guaranteeing actions and questionings from the teacher educator that cause the teachers to reflect and establishing relationships between theory, experiences and these teachers' practice (Bransford et al., 2000).

The *PLTT domain* highlights that teachers need opportunities to learn collectively and through experiences related to their own teaching practices (Ball & Cohen, 1999). PLTTs are composed of situations based on the records of practice (Ball et al., 2014) which allow teachers, for example, formulate mathematical conjectures, validate and reformulate them (Silver et al., 2007), thus contributing to the mobilization and (re)construction of knowledge necessary for teaching (Ball et al., 2008). It is also understood that PLTTs provide opportunities for teachers to develop knowledge that is central to their teaching, as they engage in tasks and activities that are the core of their daily work (Smith, 2001), within a work cycle that involves the act of planning what to teach and which tasks could provide and elucidate the mathematical knowledge to be built (Serrazina, 2017).

Finally, the *DIAP domain* draws on studies that point to collective participation (Gibbons & Cobb, 2017) of teachers and assumes that professional learning opportunities are materialized at the time of exchanges between peers and through dialogue and professional communication (Craig & Morgan, 2015) between teachers and between them and teacher educators. One approach that favors such interactions is that of exploratory teaching, as it provides collective discussions and presupposes the circulation of ideas, experiences and mathematical and didactical knowledge among teachers (Stein et al., 2008). The DIAP domain is characterized by (i) promoting mathematical and didactical discussions as a means to promote professional learning for teachers (Heyd-Metzuyanim et at., 2016); (ii) involving teachers in an environment that promotes argumentation and justification (Mata-Pereira & Ponte, 2017) when discussing mathematical tasks for students; (iii) encouraging the use of correct and appropriate mathematical language for the students' educational level (Adler & Ronda, 2014); and (iv) assisting teachers to recognize the importance of dialogical communication between themselves and their students (Craig & Morgan, 2015).

2.2. *Mathematical Knowledge for Teaching Early Algebra*

Based on the Shulman's seminal work (1986), several researchers have been working to understand and characterize the mathematical knowledge that is specific to teaching. We highlight a theoretical framework that is widespread in Brazil and other countries, the Mathematical Knowledge for Teaching (MKT) by Ball et al. (2008), which involves the mathematical knowledge necessary for the teacher to exercise their teaching mathematics role, as it is a theory based on teaching practice (Figure 2).

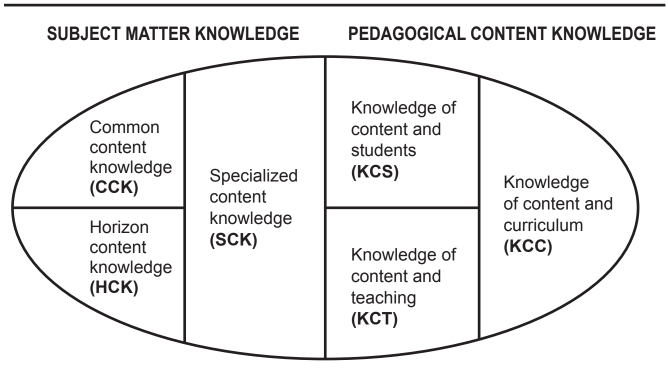


Figure 2. Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403).

In these domains, KCT combines knowledge of mathematics content with knowledge regarding your teaching at school. Meanwhile, KCS combines knowledge of mathematical content and knowledge regarding students, a kind of knowledge that "highlights the importance of understanding students' mathematics to be an effective teacher" (Sevinc & Galindo, 2022, p. 155). Finally, KCC refers to the knowledge that the teacher has regarding the presence of mathematics in the curriculum throughout the school years, and also a knowledge that allows evaluating the use of different materials/didactic resources suitable for each moment of the school period.

On the other hands, CCK is constituted by mathematical knowledge not restricted to teaching, such as, for example, recognizing a wrong answer and, SCK domain refers to mathematical knowledge that is normally not needed for purposes other than teaching. According to Ball et al. (2008), the demands of the job of teaching mathematics require a body of specialized mathematical knowledge for teaching, such as knowing different ways to solve a mathematical task. HCK is a knowledge of content that allows the teacher to have an awareness of how mathematical topics are related throughout the mathematics included in the curriculum.

Within the scope of Algebra, the works of Ball and her collaborators reverberate in the theoretical framework called Knowledge of Algebra for Teaching (KAT) (McCrory et al., 2012). McCrory et al. (2012) propose three categories for what they consider essential knowledge for an effective teaching of algebra, they are: school algebra (SA); advanced mathematics (AM); and algebra-for-teaching knowledge (ATK). SA is about mastery or proficiency in what they are going to teach; AM refers to the mathematical knowledge that teachers must have in addition to the mathematics that is directly related to teaching; and ATK, closely related to SCK, relates to the opportunities that teachers should have to learn mathematics in a way that broadens and deepens their knowledge and understanding of mathematics in a manner that specifically contributes to their teaching (McCrory et al., 2012).

McCrory et al. (2012) present three other categories, referring to the teaching practices of school algebra: *decompressing*, *trimming*, and *bridging*. *Decompressing* indicates the teacher's need to decompress their knowledge in teaching practice. For example, for elementary school teachers, the computational algorithms used in arithmetic operations, such as long division and division by a fraction, need to be unpacked. For secondary school algebra teachers, algorithms need to be decompressed to solve equations and systems of equations, to simplify expressions and to move between representations (textual, symbolic, graphic, tabular, etc.). *Trimming* means that the teacher must "trim" the mathematical content in a way that it is accessible to the student in the school year in which such content is being taught. *Bridging* means that the teacher needs to establish connections between mathematical ideas, including connections between ideas from school algebra, abstract algebra and real analysis, relating these different areas (McCrory et al., 2012).

In this way, the MKT domains proposed by Ball et al. (2008) and the three categories of teaching practices presented by McCrory et al. (2012) offer support for understanding and implementing the introduction of algebra in the first years of school. About this, studies such as those by Blanton and Kaput (2005) and Blanton (2008) point to the importance of introducing algebra in the early years of schooling, in order to develop algebraic thinking in students. Blanton and Kaput (2005) point to algebraic thinking as an important process in which students generalize mathematical ideas. Blanton (2008) highlights two key areas of algebraic thinking: (1) generalized arithmetic and (2) functional thinking. In addition, Carpenter et al. (2005) indicate the ability to look at expressions or equations in their broadest conception, revealing existing relationships and also highlighting the importance of developing in students a relational understanding of the equality symbol.

Kieran (1981) points out three different meanings for the equality symbol: operational, equivalence and relational. The operational meaning, the most worked with in elementary schools and often the only one taught in Brazilian schools (Trivilin & Ribeiro, 2015), gives the student the idea that, after the equality symbol, the result of an operation should always be included and, generally, only a single quantity is accepted as true. The second meaning of the equality symbol, that of equivalence, allows one to establish many ways of representing a number through numerical equalities, and still work the equivalence between the terms that make up numeric expressions. Finally, the last meaning of the equality symbol is the relational one, through which relationships between expressions are established, and the understanding and use of the properties of the operations (addition and multiplication) is also pointed out.

Another key area of early algebra, functional thinking is one of the major components of algebraic thinking (Cañadas et al., 2016). Para Cañadas et al. (2016, p. 4-5), "is that functions constitute a way to introduce students into algebra and should be dealt with longitudinally, beginning in the *early* elementary grades". Functional thinking "draws on a different skill set than does generalized arithmetic. It requires children to attend to change and growth" (Blanton, 2008, p. 5).

The promotion of functional thinking involves establishing a relationship between quantities, understanding how one varies from the other (Zapatera Llinares, 2018). Functional thinking "is about making elementary grades mathematics – including arithmetic – deeper and more meaningful for children" (p. 57). Thus, solving combinatorial problems, through the use of intermediate representations such as lists and possibilities tree, allow students, from early years, to develop mathematical patterns and structural relationships (Mulligan et al., 2020) and, gradually, the systematization of possibilities and the generalization from numerical expressions (such as the multiplicative principle) (Borba et al., 2021).

In Asquith et al. (2007) study, middle school teachers were asked to predict student responses to assessment items written with a focus on the equals sign. Although most knew that some students think the equals sign means "give the answer", the extent of this misconception was not accurately predicted, with teachers predicting that many more students would give a relational definition of the equals sign than would actually happened. Similarly, Vermeulen and Meyer's (2017) study on teachers' mathematical knowledge for teaching the equal sign indicated that, in general, they did not have the knowledge and skills to identify, prevent, reduce or correct conceptions students' mistakes about the equals sign.

Wilkie's (2014, 2016) studies on the professional learning of Upper Primary School teachers showed that, although a considerable part of the teachers interviewed show that they have content knowledge to develop their students' functional thinking, few have knowledge of pedagogical content. In the same direction, Pang & Sunwoo's (2022) study with elementary school teachers about their knowledge for teaching functional thinking showed a superficial understanding of its central ideas. Although many of them were able to develop mathematical tasks corresponding to simple relationships between two quantities, some of them had difficulties in developing tasks involving more complex relationships.

3. The study's methodology and context

The research was carried out in two similar Professional Development Program in Early Algebra (henceforward PD-EA1 and PD-EA2), based on the PLOT model (Ribeiro & Ponte, 2020). In both cases, the aim was to enable the participating teachers to expand their professional knowledge in order to teach Early Algebra, and had 14 face-to-face weekly work sessions of 2 hours each, around one semester. PD-EA1 performed with six PMTs (A, B, C, D, E, F) and a Teacher Educator (TE1) (Ribeiro) from the same municipal public school in São Paulo/Brazil. In PD-EA2, among the participants, one of them, teacher G, was a PMT, and four others (H, I, J, K) were prospective mathematics teachers, with experience in the elementary and high school, and a Teacher Educator (TE2) (Trevisan).

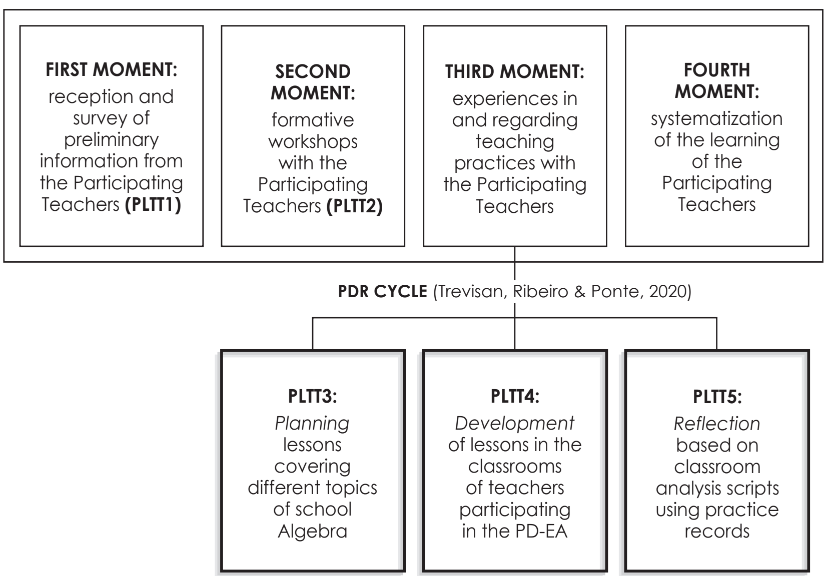


Figure 3. Structure of the Professional Development Program in Early Algebra

Each one of them was structured in four moments (Figure 3), which included the application of five PLTT. The first two of which (PLTT1 and PLTT2) were intended to survey the participating teachers' prior knowledge of Early Algebra. The last three aimed to give participating teachers the opportunity to experience an interactive cycle of planning (P), development (D) and reflection (R) of collectively prepared lessons, which covered different topics of school algebra

– PDR cycle (Trevisan et al., 2020). Stage P (PLTT3) involved the elaboration (in small groups) and collective discussion of lesson plans covering different topics of school algebra, with tasks chosen by the participating teachers themselves. Afterwards, stage D (PLTT4) took place in these teachers' classrooms. Finally, in stage R (PLTT5), the teacher educators organized analysis scripts for these classes using practice records produced in the classes held, focusing on two dimensions: the different resolution strategies used by the students, and the role and actions of the teachers in three moments of the development of the class (in the presentation of the task, in the monitoring of students' work in the groups and at the orchestrating collective discussions in the end of classes).

A characterization of PD-EA1 and PD-EA2 is presented in Figure 4.

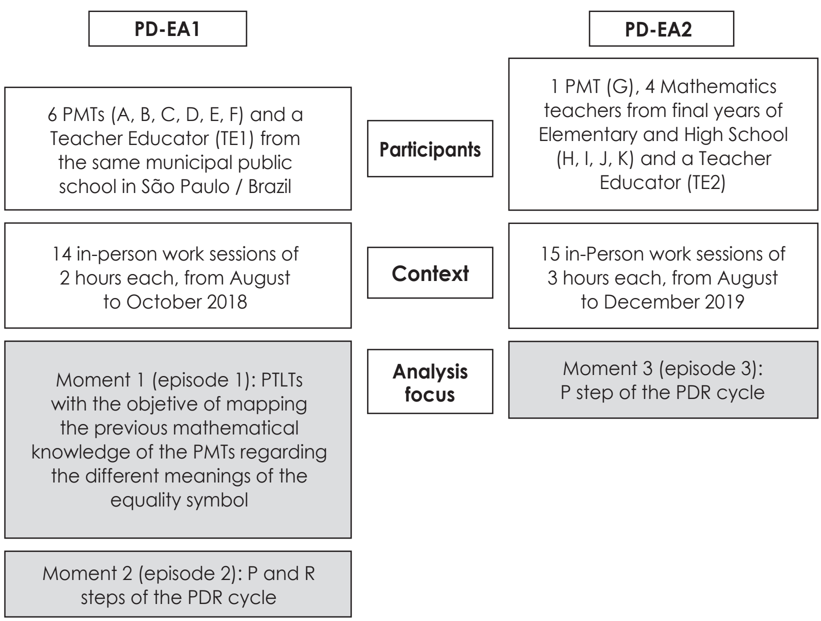


Figure 4. PD-EA Characteristics

In this paper we analyze three episodes, the first two being performed in Professional PD-EA1 and, the third, took place in PD-EA2. The first episode was enacted during the first eight sessions, when PLTT1 e PLTT2 were used in order to map the PMTs' previous mathematical knowledge about the different meanings of the equal sign. In the second episode, PLTT3, PLTT4 e PLTT5 were worked in sessions 9 to 12 in order to mobilize and (re)build mathematical and didactical knowledge of PMTs referring to Early Algebra. Specifically, this episode cover P and R stages of this cycle PDR, when discussions are presented regarding the planning of a collectively elaborated class and then, the moment in which the teacher educators transformed the practice records of the class developed in a PLTT5 of reflection on the role and action of the teacher. Finally, in the third, each participating teacher organized the lesson plan (PLTT3) that could be developed in their own class, involving functional thinking, from the choice of a mathematical task that could generate discussions among students. These teachers presented their initial version of the planning, which was discussed collaboratively with the other participants.

The methodological approach follows the principles of qualitative research in an interpretive theoretical perspective (Crotty, 1998). Data were collected through audio and video transcripts of sessions in which PLTTs were applied, as well as the written reports produced by the participating teachers. This information was grouped, creating an inventory organized by session and by data collection instrument. This procedure allowed information to be gathered, compared and analyzed. From this inventory, three moments were organized for analysis: (i) one that covers the mapping of previous mathematical knowledge of the PMTs on the meanings of the equality symbol, (ii) another that considers the planning and reflection of a lesson on the meanings of equality symbol, and (iii) one that involves planning a class encompassing functional thinking.

To carry out the analyses, special attention was given to the emergence of PLOTs when teachers collectively planned and analyzed lessons involving different meanings of the equality symbol and the development of functional thinking. More specifically, we sought to investigate the mathematical knowledge for teaching early algebra mobilized by teachers when they experience collective opportunities for professional learning. Professional learning opportunities were identified based on the PLTTs used, the DIAP and the RATE (Ribeiro & Ponte, 2020). On the other hand, the MKT domains (Ball et al., 2008) were identified from discussions regarding the different meanings of the equality symbol (Kieran, 1981) and the manifestations of development of functional thinking (Blanton & Kaput, 2008).

4. Findings

4.1. *Episode 1 – Exploring the Meanings of Equality*

In the meeting dedicated to survey the PMTs' prior knowledge regarding the different meanings of the equality symbol (equivalence and relationship), the two groups analyzed the mathematical task (Figure 5) and, subsequently, the students' resolutions about the task (Figure 6), focusing on students' challenges and resolution strategies.

Part of PLTT 1 - The Equality Symbol

Teacher Jane was analyzing her 4th grade student´s answers to the proposed task.

Arthur and his sister Cecilia received from their aunt the same amount of money. Arthur decided to save 20 reais in his piggy bank and to keep some money to take to school. Cecilia saved 16 reais in her piggy bank and put aside the rest to buy some stickers.

As the two children received the same amount of money, we can establish the equality:

20 + ____ = 16 + ____

i. Determine the amount of money each child put aside to be spent.
ii. Explain how you reached the result.

Figure 5. Mathematical task. Adapted from Barboza (2019, p. 88)

Resolution 1

Arthur took 10 reais to school and Cecilia put aside 14 reais to buy herself stickers. This is how we thought this through, if they received the same amount of money, then Cecilia spent 4 reais more than her brother and we think that he took 10 reais, because you can only take 10 reais maximum to school. So we save 20 + 10 + 30 and 16 + 14 = 30.

Resolution 2

Arthur took 36 reais to school and Cecilia spent the same 36 reais on stickers, because they had the same amount of money. We got to that answer adding up the two numbers that were in the calculation: 20 + 16.

Resolution 3

Arthur took 5 reais to school and his sister spent 9 reais on stickers. We think that, if the two of them received the same amount of money and he saved 4 reais more on his piggy bank, then Cecilia had 5 reais plus 4 reais to spent on stickers. We got to this answer by doing 20 + 5 = 16 + 9, because Arthur saved 4 reais more than his sister.

Figure 6. Real and Fictitious Student Resolutions. Adapted from Barboza (2019, p. 89)

In the group of PTMs A, B and C, they all managed to solve the task, but they could not expand the SCK and kept numbers restricted to the solution of the task [1.2], so TE1 encourages them to think about other possible solutions [1.3] in an attempt to extend the SCK. The PTMs, from the action of TE1 expand their SCK to other resolution possibilities [1.5], [1.6] and [1.7], but they do not expand the SCK in the sense of using the equality symbol with a meaning other than the operational one.

[1.1] TE1: You put 10 and 14; 4 and 8; ... what relationship exists between each pair of numbers?
[1.2] A: That we only work with even numbers?
[1.3] TE1: But can you only put an even number? What if I put 15?
[1.4] A: Possible too.
[1.5] B: 20+15 equals 35.
[1.6] A: So, there would have to be something on the other side to equal 35 too. 16 plus?
[1.7] B: 19.

TE1, realizing that the PMTs only used the equality symbol in the operational sense, began to be instigating them with questions, with the intention that they would perceive the relationship between their answers, noting the meaning of equivalence [1.8] and [1.10], thus expanding their SCK.

[1.8] TE1: Everything you calculated on one side of the symbol, you tried to find the balance by calculating on the other side.
[1.9] A: Yes, we added to find the balance on the other side.
[1.10] TE1: Yes, but is there a way for us to determine the value to be placed on the other side, without having to add each side separately? […] Notice, you did 20+10, and on the other side of the equality there would be 16 plus?
[1.11] B: 14.
[1.12] TE1: Here, they put 20 + 4, and on the other side there would be 16 plus?
[1.13] A: 8.
[1.14] TE1: And then?
[1.15] A: Oh, it's always adding 4!
[1.16] TE1: And why?
[1.17] B: Wow, I hadn't realized that.
[1.18] A: Only now I noticed that, and why?
[1.19] C: Oh, it's because between these two [20 and 16] there is this difference of 4.
[1.20] TE1: And if there is 4 less here... on the other side...
[1.21] C: On the other side there will be 4 more.

With this discussion, we realize that the PMTs recognize the regularity existing in the equality, thus moving from the operational meaning [1.5] and [1.6] and starting to also perceive the equivalence meaning of the equality symbol [1.15], [1.19], [1.21]. It is noteworthy how important the action of TE1 was, instigating the discussion to create opportunities for the expansion of SCK of the PMTs, who start to wonder if they could generalize the pattern found to any other tasks:

[1.22] A: And it will always be 4? In any task?
[1.23] TE1: In this task, yes. But what if the boy had saved 15 and the sister 10, would it be the same?
[1.24] A: 15 and 10... Then it would be 5. Is that it?
[1.25] TE1: Exactly.
[1.26] B: Oh. It's from here!
[1.27] TE1: Yes, it is the relationship that is established, since the two received the same values; so, if here there is 5 more and the other 5 less, to maintain the equivalence I have to consider this.
[1.28] B: Wow, look at this, if I put 15+5, just put 10+10, the difference really is 5.

Thus, TE1 leads them to conclude that one can look at the equality symbol with the meaning of equivalence and even relation. Thus, it can be seen, with the end of the discussion, that both the mathematical task and the action of TE1 during the discussions provided the PMTs with professional learning opportunities in order to identify the equivalence and relational meaning of the equality symbol, expanding their SCK.

Next, the PMTs began to analyze the students' resolutions (Figure 6), using the different meanings of the equality symbol that were mobilized previously, in order to reflect on the students' resolutions. The group formed by D, E and F made their first analyses and conjectures:

[1.29] D: But then in this case here they didn't notice [resolution 1].
[1.30] E: The equality.
[1.31] D: Yes. He ignored the equality symbol as equivalence.
[1.32] F: So, let's go back here [re-read resolution 1].
[1.33] E: They got it. Not only did they perceive equality, they also found the equivalence. And he also realizes that Cecilia has a difference of 4 reais.
[1.34] D: Different from this one then [resolution 2]?
[1.35] E: Very different, because this group [resolution 1] perceives equality, and this one adds everything up [resolution 2].
[1.36] M: Look, that's right [resolution 3]. They understand the reasoning and still discover the difference of 4 reais.

From the discussions, we see that the PMTs are using the equivalence meaning of the equality symbol to reflect on the students' resolutions [1.31], [1.33], [1.35] and [1.36]. Based on the discussions presented in this Episode, we can state that PLTT, together with the performance of TE1, allowed the teachers to mobilize and expand their mathematical knowledge to teach the different meanings of the equality symbol – in particular, by expanding their understanding of the meanings of the equality symbol, from operator to equivalence [1.19], [1.21], [1.31], [1.33], [1.35] and [1.36].

4.2. *Episode 2 – A lesson on the meanings of the equality symbol: from planning to reflection*

During the meeting dedicated to lesson planning, the two groups of PMTs began the work by analyzing mathematical task (Figure 7), focusing on the challenges and proposals for students and evaluating how they could work on them in the classroom.

THE BOWLING GAME

The 5th grade classroom A student were divided in 4 teams to play a game of bowling. Look at the scoreboard Teacher Valter kept of the rounds:

	ROUND 1	ROUND 2	ROUND 3	TOTAL
TEAM 1	12	13	15	
TEAM 2	15	7		35
TEAM 3		15	8	35
TEAM 4		10		

Some of the data was not written down by the Teacher. Answer the questions and help complete the scoreboard:

a) With how many points did Team 1 finish? Explain how you got to that result.
b) How many points did Team 2 have on round 3? Explain how you got to that result.
c) How many points did Team 3 have on round 1? Explain how you got to that result.
d) Team 4 had, in total, only half the points of Team 1. Determine the total of points Team 4 had in rounds 1 and 3.

Figure 7. Mathematical task chosen for lesson planning. Adapted from Barboza (2019, p. 91)

The groups A, B, and C initially anticipated possible resolutions that students might make use of, and difficulties they might face in carrying out the task:

[2.1] B: In item (d) they are the ones who will determine. Each one can have their possibilities.
[2.2] A: Yes, each one will distribute as they see fit, as long as they reach 20 in total.
[2.3] TE1: Are they used to having more than one right result?
[2.4] B: No.
[2.5] A: No, because they know they can have multiple strategies to achieve one answer, one result. But several right results...
[2.6] TE1: And do you think this is extra challenging?
[2.7] A: I think it's extra challenging, yes.

We can observe that the PMTs mobilized PCK, specifically in regard to KCS, when discussing the strategies that the students would possibly use [2.1] and [2.2], and KCT, especially because they reflected on the challenge that the task was posing [2.7]. Noting that the discussions of the PMTs focused on the knowledge of students and teaching, and with the aim of proposing reflections on the meaning of equivalence of the equality symbol, TE1 introduces a question:

[2.8] TE1: And do you think that, in this question or any other, they would be able to look and establish equivalence relations.
[2.9] B: Yes.
[2.10] TE1: And, why? Elaborate.
[2.11] B: Because they will realize that...
[2.12] A: You can add different digits and get the same result.

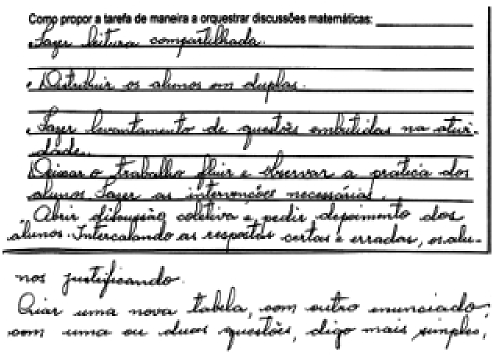
We recognize that the teachers expanded their meanings of the equality symbol, now perceiving it with the sense of equivalence [2.9]. This learning opportunity arises as a result of the intervention of TE1 [2.8], who proposed a question that would help the PMTs to move from discussions more focused on pedagogical knowledge to a discussion of a more mathematical nature. We can also observe that they mobilize knowledge related to SCK [2.11] and [2.12], by expanding their knowledge regarding the meanings of the equality symbol and relating them to teaching.

Still wanting to promote new reflections on the meaning of equivalence of the equality symbol, TE1 [2.13] introduced another question (bringing light to the possibilities of relationships that could be established between the missing data in the table – Figure 7):

[2.13] TE1: Look at Team 2 and Team 3, what do they have in common?
[2.14] A: The same result.
[2.15] TE1: And in the rounds, see if they have any relationship.
[2.16] A: Um, they’re both 15.
[2.17] TE1: And in the other round, one has 7 and the other has 8. Could it be that observing this is a way of looking at what will be the relationship between these two spaces to complete?
[2.18] C: I don’t think so, I think they’ll just calculate it.
[2.19] A: I find it difficult to look at this. Because I think they will add up the installments, which is their practice, and then they will take the smaller portion from the larger one to find the unknown value.

The discussions made possible to the PMTs, because they are working in groups, together with the teacher educator’s interventions, seem to us to generate learning opportunities so that they begin to indicate different possibilities for resolution [2.18] and begin to observe possibilities of establishing relationships in the task between the members of an equality [2.14] and [2.16].

Something else to be highlighted refers to the opportunities that the PMTs experienced regarding curricular knowledge and teaching of early algebra. In view of the agreement on the use of the mathematical task to be used in teacher A's classroom, the six PMTs established which objectives can be developed in the lesson in question (Figure 8), and for that, they experienced the analysis, in group, of the BNCC, thus being able to know and explore more deeply the thematic unit "Algebra", recently introduced in Brazil (Brasil, 2017). It can be seen from the rationales that the PTMs sought to create contexts for the classroom, which would allow interactions in regard to the content to be worked on (KCT) as well as allow students to advance in their learning (KCS).


Como propor a tarefa de maneira a orquestrar discussões matemáticas:
Fazer leitura compartilhada.
Distribuir os alunos em duplas.
Fazer levantamento de questões embutidas na atividade.
Deixar o trabalho fluir e observar a prática dos alunos. Fazer as intervenções necessárias.
Abrir discussão coletiva e pedir depoimento dos alunos. Intercalando as respostas certas e erradas, e os alunos justificando.
Criar uma nova tabela, com outro enunciado, com uma ou duas questões, digo mais simples;

How to propose the task in order to orchestrate mathematical discussions:

- Share readings;
- Pair up the students;
- Raise questions embedded in the activity;
- Let the work flow and observe student practice. Make necessary interventions;
- Open collective discussion and ask for testimony from students, merging and explaining right and wrong answers, justifying them;
- Create a new table, with another question, with one or two questions, simpler.

Figure 8. Protocol used to justify the chosen mathematical task (Barboza, 2019, p. 109)

During the analyses, the PMTs were satisfied with the students' involvement with the task and, especially, with the expression used by one of them, "it added up to the same equality". They also signaled the importance of the teacher taking advantage of moments of discussion among students, to establish mathematical connections and systematize concepts [2.25], [2.26] and [2.28]:

[2.25] D: Wow, isn't that when you can close the concept you want to work on?
[2.26] A: Ah, the one of equivalence!
[2.27] D: Yes.
[2.28] E: It is important to systematize it with the mathematical language.

The PLTT included records of practice containing students' mathematical strategies and reasoning about the equivalence meaning of the equality symbol. One of the students explained: "We realized that, in Team 2, it was 15 first. And then in Team 3, it was 15 too. And the difference is 1." Another added: "They both have the same result, and since you're saying here it's 35, they're equal. I could put 1 more here, like, we could have 13 here, but it's 12, and here it could be 12, but it's 13. It's just that in this one [pointing to 8], there's one more than this one [pointing to 7]."

Encouraged by the possibility of analyzing episodes that occurred during the lesson, one of the PMTs stated: "Wonderful, I liked it. They showed that they understood it, yes [the equivalence relation of the equality symbol]." If on the one hand it is possible to consider that the PMTs realized the mathematical strategies adopted by the students, on the other hand, it caught our attention that, during the lesson planning, the teachers believed that the students would not pay attention to the equivalence, or the relationship between the equalities that appeared in the mathematical task. Therefore, we understand that TE1's choices and actions, combined with the PLTT's design of reflection, ended up providing learning opportunities in which the PMTs (i) identified a non-trivial solution to the task, (ii) reorganized their professional knowledge to the use of unusual tasks and strategies, (iii) challenged students to engage in productive mathematical discussions.

4.3. *Episode 3 - Functional Thinking in Elementary School*

In the meeting dedicated to PTLT that involved the planning of the lesson, PMT G presented the mathematical task that she had prepared for her 4th grade class (Figure 9), as well as the anticipation of four possible ways to solve it (Figure 10). A discussion led by TE2 was then conducted, evaluating how the task could be worked in the classroom, as well as its potential for the development of students' functional thinking.

In a basketball tournament, the teams of the following states got to the final: Paraná (PR), Acre (AC), Paraiba (PB) and São Paulo (SP). In how many different ways can the podium be composed?

Figure 9. Mathematical task proposed in PMT G planning

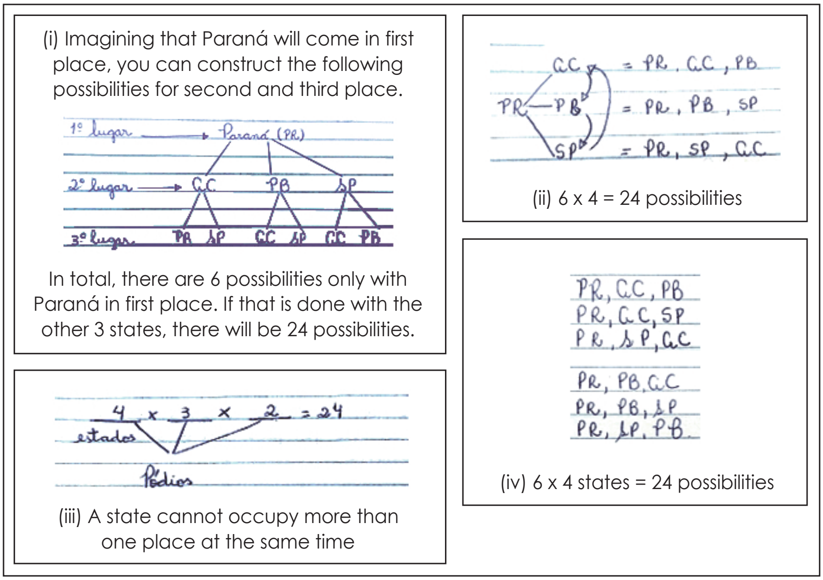


Figure 10. Anticipations of possible ways to solve the task made by PMT G

In a first interaction, TE2 asks G about her objective with the task.

[3.1] G: I want him to understand that it's a combination, but I don't know if it involves functional thinking. He has to make the pairs, find out who can be first, second and third. How many possibilities will he have in here? But I couldn't generalize a calculation.

In this excerpt, G manifests two aspects that permeated the discussion: the doubt whether the task allows the development of functional thinking, and the difficulty in seeing some kind of generalization, reaching an algebraic expression, which G calls "calculation". The teacher educator tries to highlight the presence of functional thinking.

[3.2] TE2: You can try to help the student figure out how to determine the total without writing them down one by one.

[3.3] G: So that's what I couldn't see. A calculation.

In order to mobilize some aspects of KCS, leading teachers to recognize the importance of communication between them and their students, in this case, for the development of functional thinking, the teacher educator points out the importance of helping the student to discover the total number of possibilities, without necessarily listing all cases [3.2]. G, in turn, explains once again that she was not able to “see a calculation” [3.3], as she cannot recognize the task’s potential for generalization, which suggests the need to deepen her SCK. The action of TE2 is, then, in the sense of helping her understand how to reach a generalization, based on a suggestion made by H, to reformulate the task starting with a smaller number of states.

[3.4] TE2: For three states, PR, AC and PB, in how many different ways can the podium be composed?
[3.5] I: Six ways.
[3.6] TE2: What would they be?
[3.7] I: $3 \times 2 \times 1$.
[3.8] G: If it’s only three. Wait a minute... [G writes on the blackboard and fixes PR in first, followed by AC and PB. Then she writes PR, PB and AC]. It will give you two, only two possibilities only.

If, on the one hand, Teacher I, who knows and teaches the multiplicative principle for high school students, appears to have this notion in a mechanical way, as something memorized [3.5], [3.7], on the other hand, Teacher G goes to the blackboard and uses the possibilities tree as a resource to determine the number of possibilities [3.8]. Here, aspects of the SCK can be noted in terms of the difference in the understanding of the task by these two teachers, possibly due to the context in which each one works. HCK is also evidenced to the extent that, on the one hand, the PMT does not know how to deal with a possible generalization (which would be given, for example, by the idea of arrangement) and, on the other hand, high school teachers do not seem to recognize more “introductory” forms of teaching, preferring a more automated approaches (for example, making use of the possibilities tree and leading to a generalization). Other issues related to KCS and KCC are brought up during the discussion.

[3.9] J: Teacher G, do you think your students would reach this multiplication?
[3.10] G: I think so, with some help.
[3.11] J: Because I find it very difficult for them to recognize multiplication.
[3.12] I: This is a multiplicative principle, but it is only taught in the 2nd year of high school.
[3.13] TE2: No, it is already taught in 4th year.
[3.14] I: No, it’s in 7th grade that we teach this, actually. Matching clothes.
[3.15] G: No, in 4th grade you combine clothes, you combine juice with a snack.

Although Teacher G considers that her students are capable of solving the task (albeit with some intervention) [3.10], J still considers it very difficult for this level of education [3.11]. Furthermore, G [3.13] and TE2 [3.13] highlight that this content is already explored in earlier years, a fact unknown by I [3.14].

As mentioned by G [3.15], the BNCC indicates that for the 4th year, the student is expected to solve, with the support of images and/or manipulative material, simple counting problems, such as determining the number of clusters possible when combining each element of one collection with all the elements of another, using personal strategies and forms of recording (Brasil, 2017). PMT G mobilizes knowledge related to KCC and allows the other teachers in the group to learn about the elementary school curriculum, as they do not have experience at that school level.

Assuming the importance of knowing different ways to solve the mathematical task and understanding the mathematical connections that can be established from them, TE2 circles back to the original task proposed by G (Figure 10), seeking to help teachers recognize a generalization.

[3.16] TE2: With three Sates, there are 6 options. And with four States?
[3.17] I: It's 4 × 3 × 2, it's 24.
[3.18] G: It's 24.
[3.19] TE2: This result, teacher G, didn't you anticipate it?
[3.20] G: This one, look! [Figure 11]. But I couldn't see it on the calculation.

On the one hand, Teacher I refers again to the multiplicative principle, privileging the algorithm, demanding opportunities to give new meaning to their SCK, in particular expanding and deepening their knowledge and understanding of mathematics, in a way that serves to rethink the way they teach. On the other hand, G does not seem to understand why multiplication (although she had used it in one of the strategies she anticipated), an aspect of CCK, and needs to explore the knowledge necessary for teaching Algebra in early years, systematizing the possibilities and generalizing. In order to meet these demands and create new learning opportunities, TE2 again intervenes, and the discussion continues:

[3.17] TE2: With three states, putting PR first, what can I have in second?
[3.18] All: AC and PB.
[3.19] TE2: There are two options. And how many states can I put first?
[3.20] All: Three.
[3.21] TE2: So, the total is 2 + 2 + 2, or, 3 × 2. If there are 4 states, there are six podium options for each state, as there are four states, they will be 6 + 6 + 6 + 6 or 4 × 6.
[3.22] H: But you can do 4×3×2 too.
[3.23] TE2: I don't know if it's natural for students to multiply these numbers. What if there are five states?
[3.24] G: If he thinks the options for one, and then adds it up, he gets it.

[3.25] TE2: Anyway, if he's able to not need to write all the possibilities, even if he writes it for one state or two, he's already had some kind of generalization.

In this final part of the discussion, TE2 seeks to differentiate two resolution strategies that can be used to solve the task. One of them is the explicit use of the multiplicative principle, a possibility he understands to be difficult for students in early grades. Another is the use of one-to-many correspondence. In order to help the group to systematize some kind of generalization, and thus develop aspects of functional thinking, TE2 suggests that, in the case of five States, one can initially think about the number of possibilities for one of them fixed (by using a scheme, for example), and then add them up. From there, G [3.24] highlights that it is possible for students to reach the total of possibilities, if they are able to think of using one-to-many correspondence.

The interaction between TE2 and teachers allowed participants to refine and expand their SCK. In the specific case of PMT G, we understand that the actions of TE2, combined with the reflection design of PLTT3, provided learning opportunities when: (i) they offered the teacher a space to reflect on the "calculation" she had been searching for since the beginning [3.1 and 3.3]; (ii) made it possible to reflect on what a generalization could be, considering that it would not necessarily need to involve all possibilities (as repeated in I [3.7 and 3.17]) or involve a letter to characterize it as a generalization. As TE2 tried to discuss [3.21, 3.23 and 3.25], the "calculation" sought could involve a step before the idea of $4 \times 3 \times 2$ and, even so, the generalization would be present; (iii) it allowed her to reorganize her professional knowledge to use the task she proposed, deepening her understanding of the different strategies she anticipated (Figure 11).

5. Discussion of the results

In this section, we seek to highlight the results observed in each of the three episodes, relating them to each other and to the literature and theoretical framework chosen to support the identified results, especially regarding to *Opportunities to Learn about Mathematical Knowledge for Teaching Early Algebra*. To understand what knowledge teachers mobilized and built during formative processes, we consider reflections on the different meanings of the equality symbol, as well as manifestations of functional thinking development along PD-EA1 and PD-EA2.

The format chosen by TE1 and TE2 for the organization of the PLTTs made it possible to expand CK, with evidence identified throughout the three episodes, especially by the RATE (Ribeiro & Ponte 2020) that supported the PMTs to think of different strategies for solving the tasks (Silver et al., 2007), thus expanding the potential and possibilities of developing their own algebraic thinking (Blanton & Kaput, 2005), leading them to expand their SCK, more specifically ATK (McCrory et al., 2012).

In Episodes 1 and 2, both the mathematical tasks and the actions of TE1 during the discussions (Heyd-Metzuyanim et al., 2016) were to use the equality symbol with meanings other than the operational one (Kieran, 1981), enabling the teachers identify equivalence and relational meanings. In Episode 2, the teachers were given the opportunity to reflect on decompressing this knowledge in the teaching practice of teacher A (McCrory et al., 2012), reflecting on how students socialized their strategies, expanding their meanings of the equality symbol to include the key idea of equivalence (Kieran, 1981).

In Episode 3, the action of TE2 was intended to help teachers (Desimone, 2009), especially PMT G, to develop mathematical patterns and structural relationships from a combinatorial problem (Mulligan et al., 2020) and obtain a generalization from a numerical expression (Borba et al., 2021). This Episode evidenced HCK, with TE2 problematizing the different understandings of the task by the teachers (Mata-Pereira & Ponte, 2017), allowing G to reflect on a possible generalization (a "calculation" that she was looking for from the beginning) as a manifestation of functional thinking (Blanton & Kaput, 2005). For other teachers, the kind of discussions organized (Adler & Ronda, 2014) by TE2 offered opportunities of trimming the task (McCrory et al., 2012), reformulating them in more accessible ways, both for early grade students and for those of more advanced levels, articulating the use of the possibilities tree and carrying out a multiplication as a representation of the generalization of these possibilities (Borba et al., 2021).

Also in Episode 3, in order to expand CCK, more specifically SA (McCrory et al., 2012), TE2 sought to differentiate two resolution strategies that can be used to solve the combinatorial problem: one of them explicitly using the multiplicative principle and, another, more primary and natural, with the one-to-many correspondence (Montenegro et al., 2021). This is essential knowledge for PMTs to mobilize and (re)structure in order to give effect the recent inclusion of Algebra in the curriculum of early year students in Brazil.

In each of the three episodes, there were opportunities for teachers to mobilize and expand their KCS, when discussing the strategies that students would possibly use to solve the tasks. It is noteworthy that, in both contexts, teachers considered the mathematical task very difficult for the early years, "underestimating" the ability of students to engage in more exploratory teaching situations (Canavarro et al., 2012). In Episode 2, during the analysis of student discussions, the PMTs were surprised by the involvement of students and their resolution strategies, noticing the mathematical strategies adopted by them associated with the meaning of equality as equivalence (Kieran, 1981).

In Episode 3, there is a moment of discussion in which TE2 sought to lead teachers to recognize the importance of communication between them and their students (Craig & Morgan, 2015), helping them to discover the total number of possibilities, without writing case by case (Mulligan et al., 2020). This perception that, when students do not need to write down all the possibilities, counting one by one to solve the task, is already a type of generalization, even if still at an initial level, and that allowed PMT G to review aspects of her practice, giving her more confidence to work generalization in the classroom, ATK (McCrory et al., 2012).

Other moments in the collective discussions promoted by the PLTT brought to light the mobilization and (re)construction of other dimensions of teachers' pedagogical knowledge. This occurred in all three episodes and was evidenced by the opportunities for teachers to rebuild their KCT. In Episodes 1 and 2, for example, there are moments in which the PMTs reflected possibilities of articulation between the discussed tasks and the Algebra thematic unit in the new Brazilian curriculum document. In Episode 2, the PMTs reflected on goals that could be developed in class with Teacher A's students. Thus, the PMTs were given the opportunity to trim the Algebra concepts involved therein, in a way that such concepts were accessible to students in early years (McCrory et al., 2012). Also in this Episode, during the analyses carried out by the PMTs on the lessons delivered, PMTs were able to recognize the importance of the teacher taking advantage of moments of discussion of student resolutions to establish mathematical connections and systematize concepts (Putnam & Borko, 2000; Stein et al., 2008).

Finally, aspects of KCC were present in Episode 3, in moments of collective discussion (Adler & Ronda, 2014) in which teachers could reflect on what reformulations the task could receive, so that it could become more suitable for early year students, or even, when teachers need to think about combinatorial problems in this level of education, a new experience for one of the participants, Teacher I.

6. Conclusions and final considerations

Aiming to understand how learning opportunities are constituted and developed so that primary teachers can approach early algebra in the elementary school, this article presents the results of a research program carried out in Brazil in the last five years. Two contexts were chosen to exemplify such results: (i) one focusing on the different meanings of the equality symbol and (ii) other focusing on generalization processes and the construction of functional thinking. The results allowed us to answer the two research questions RQ1 and RQ2, showing us how PLOT model (Ribeiro & Ponte, 2020) enabled teachers to broaden and deepen their professional knowledge to teach (Ball et al., 2008; McCrory et al., 2012) early algebra in early years of schooling (Kieran, 1981; Blanton & Kaput, 2005).

With regard to the RQ1, the results show that the PMTs were able to reflect on the decompressing (McCrory et al., 2012) of knowledge related to the different meanings of the equality symbol (Kieran, 1981; Trivilin & Ribeiro, 2015), when they discussed the lesson taught by Teacher A, namely about how students socialized their resolution strategies and expanded their meanings of the equality symbol, starting to recognize it also by its equivalence meaning. We also observed PLOTs from the analysis and reflection of the lesson taught by PMT G, reviewing aspects of KCC, especially when teachers noticed the reformulations that the task could receive (Adler & Ronda, 2014), so that they could become more appropriate for early years students, and the potential that working with combinatorial problems at this level of education has for the development of functional thinking (Blanton & Kaput, 2008).

Regarding the RQ2, the results indicate that PMTs experienced collective learning opportunities, especially through RATE (Zaslavsky, 2008) in the conduct of PD-EA1 and PD-EA2, favoring the rupture of the isolation that teachers normally face in their schools (Ball & Cohen, 1999). It was noted that PMTs expanded their SCK, especially ATK (McCrory et al., 2012), as they were encouraged to think of different task solving strategies (Silver et al., 2007) and to become aware of the potentialities and possibilities of developing their own algebraic thinking (Blanton & Kaput, 2008). Also as a result of collective learning opportunities, it was noted that the format adopted for PLTTs (Smith, 2001), prepared by TE1 and TE2, using records of practice (Ball et al., 2014), provided to the PMTs the reconstruction of their KCT, highlighting the fact that they recognize possibilities of articulation between the tasks discussed in regards to the equality symbol, and the Algebra thematic unit in the new Brazilian curriculum document, the BNCC (Brasil, 2017).

Adopting approaches in formative processes with in-service teachers, such as the one we present in this article, offers them learning opportunities that encompass moments to validate and rethink the choices and decisions they make when planning and enacting lessons, and how the collective analysis of these lessons avors the mobilization and redefinition of mathematical knowledge for the teaching of early algebra. Advances in the different dimensions of teachers' professional knowledge seem to have been enhanced by the design of the teacher education program, anchored in the PLOT model (Ribeiro & Ponte, 2020), favoring collective moments for teachers to plan, develop and reflect on mathematics lessons regarding the different meanings of the equality symbol and about functional thinking.

The results of our research program have important implications for teacher education and the professional development of primary teachers, as they show the potential of collective work, involving teachers and facilitators, mediated by professional tasks conceived from and for classroom practices involving early algebra, especially because teachers usually do not usually have an opportunity to experience and study these topics in their own schools.

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