Proof and Explanation from a Semiotical Point of View

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Man sah, dass der Austauschprozess der Waren widersprechende und einander ausschliessende Beziehungen beinhaltet. Die Entwicklung der Ware hebt diese Widersprüche nicht auf, schafft aber die Form, worin sie sich bewegen können. Dies ist überhaupt die Methode, wodurch sich wirkliche Widersprüche lösen. Es ist z.B. ein Widerspruch, dass ein Körper beständig in einen anderen fällt und ebenso beständig von ihm wegflieht. Die Ellipse ist eine der Bewegungsformen, worin dieser Widerspruch sich ebensosehr verwirklicht als löst. K. Marx, Das Kapital, Band I, p.118f [2]

RESUMEN

Una distinción entre pruebas que prueban y pruebas que explican es parte invariable de las discusiones recientes en epistemología y en educación matemática. Esta distinción se remonta a la época de los matemáticos que, como Bolzano o Dedekind, intentaron restablecer a las matemáticas puras como una ciencia puramente conceptual y analítica. Estas tentativas reclamaron, en particular, una eliminación completa de los aspectos intuitivos o perceptivos de la actividad matemática, sosteniendo que se debe distinguir de forma rigurosa entre el concepto y sus representaciones. Utilizando una aproximación semiótica que refuta una separación entre idea y símbolo, sostenemos que las matemáticas no tienen explicaciones en un sentido fundamental. Explicar es algo así como exhibir el sentido de alguna cosa. Los matemáticos no tienen, sin embargo, como vamos aquí a intentar demostrarlo, sentido preciso, ni en el sentido intra-teórico estructural, ni en comparación con la objetividad intuitiva. Los signos y el sentido son procesos, como vamos a sostenerlo inspirándonos de Peirce.

- **PALABRAS CLAVE:** Peirce, Bolzano, Semiosis, Prueba, Explicación.

ABSTRACT

A distinction between proofs that prove and proofs that explain has over and again played an important role within recent discussions in epistemology and mathematics education. The distinction goes back to scholars who, like Bolzano or Dedekind, have tried to

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2 We saw that the exchange of commodities implies contradictory and mutually exclusive conditions. The differentiation of commodities into commodities and money does not sweep away these inconsistencies, but develops a *modus vivendi,* a form in which they can exist side by side. This is generally the way in which real contradictions are reconciled. For instance, it is a contradiction to depict one body as constantly falling towards another, and as, at the same time, constantly flying away from it. The **ellipse** is a form of motion which, while allowing this contradiction to go on, at the same time reconciles it. Karl Marx (1906), *Capital*, vol I. chapter 3.

reestablish pure mathematics as a purely conceptual and analytical science. These endeavors did in particular argue in favor of a complete elimination of intuitive or perceptual aspects from mathematical activity, arguing that one has to rigorously distinguish between a concept and its representations. Using a semiotical approach which negates such a separation between idea and symbol, we shall argue that mathematics has no explanations in a foundational sense. To explain amounts to exhibiting the meaning of something. Mathematics has, however, as we shall try to show, no definite meanings, neither in the structural intra-theoretical sense nor with respect to intuitive objectivity. Signs and meanings are processes, as we shall argue along with Peirce.

• **KEY WORDS:** Peirce, Bolzano, Semiosis, Proof, Explanation.

RESUMO

Uma distinção entre provas que demonstram e provas que explicam é parte invariável das discussões recentes na epistemologia e em educação matemática. Esta distinção se remonta à época dos matemáticos que, como Bolzano o Dedekind, tentaram divisão da matemática pura como uma ciência puramente conceptual e analítica. Estas tentativas reclamaram, em particular, uma eliminação completa de os aspectos intuitivos ou perceptivos da atividade matemática, sustentando que se deve distinguir de forma rigorosa entre o conceito e suas representações. Utilizando uma aproximação semiótica que refuta uma separação entre idéia e símbolo, sustentamos que a matemática não tem explicações em um sentido fundamental. Explicar é algo assim como exibir o sentido de alguma coisa. Os matemáticos não têm, contudo, como vamos aqui a intentar demonstrar, sentido preciso, nem o sentido intra-teórico estrutural, nem comparação com a objetividade intuitiva. Os signos e o sentido são processos, como vamos a sustentar inspirados em Peirce.

• **PALAVRAS CHAVES:** Peirce, Bolzano, Semiótica, Prova, Explicação.

RÉSUMÉ

Une distinction entre preuves qui prouvent et preuves qui expliquent est une partie invariable des discussions récentes en épistémologie et en éducation mathématique. Cette distinction remonte à l'époque des mathématiciens qui, comme Bolzano ou Dedekind, ont tenté de rétablir les mathématiques pures comme une science purement conceptuelle et analytique. Ces tentatives ont réclamé en particulier une élimination complète des aspects intuitifs ou perceptuels de l'activité mathématique en soutenant qu'on doit distinguer de façon rigoureuse entre le concept et ses représentations. En utilisant une approche sémiotique qui réfute une telle séparation entre idée et symbole, nous allons soutenir que les mathématiques n'ont pas d'explications dans un sens fondamental. Expliquer revient à exhiber le sens de quelque chose. Les mathématiques

n'ont pas cependant, comme nous allons tenter de le montrer, de sens précis, ni dans le sens intra-théorique structurel, ni par rapport à l'objectivité intuitive. Signes et sens sont des processus, comme nous allons soutenir en nous inspirant de Peirce.

● **MOTS CLÉS:** Peirce, Bolzano, Sémiosis, Preuve, Explication.

Introduction

Before we can address the issue of proof and explanation we have to get rid of traditional *Bewusstseinsphilosophie* (philosophy of consciousness), that is, popularly speaking, the belief that "meanings are in the head" and knowledge is some sort of mental experience. After Kant epistemology began to ramify and various new philosophies of mathematics arose in which meaning, rather than mind played the central role. But the view that there exists an epistemologically autarkic or self-sufficient epistemic subject, which serves itself from external sensations and internal experiences or representations (Vorstellungen) to thereby constitute true knowledge, is a myth and should also be abandoned.

In Part I of this paper we try to provide some pertinent arguments to this end, based on Peirce's semiotics. "Consciousness is used to denote the *I think*, the unity of thought; but the unity of thought is nothing but the unity of symbolization" (Peirce CP 7.585). Part II treats the questions of proof and explanation with respect to the ideas of Bolzano on the one hand and Peirce on the other. Part III presents some examples and tries to make a connection with current debates about the issue in mathematical education and cognitive psychology.

I. To try to understand cognition and knowledge as semiotic processes we begin by conceiving of cognition as the result of a dialectical contradiction between cognitive subject and objective reality. We feel or perceive something, but cannot turn it into cognition without a symbol and it thus remains as a mere non-categorized sensation or intuition. Or, differently: somebody might understand the logic of an argument without seeing how it applies in a particular situation and thus does not really follow it. It is futile and fruitless, for example, to expect that the object of investigation would finally reveal itself to us in plain clearness such that knowing would then amount to reading off its relevant properties.

The symbol is to mediate between conscious feeling and objective reaction and should provide this interaction with a certain form or representation. This is the only manner in which we can know, that is, by constructing a relevant representation of some kind. "A representation is that character of a thing by virtue of which, for the production of a certain mental effect, it may stand in place of another thing. The thing having this character I term a representamen, the mental effect, or thought, its interpretant, the thing for which it stands, its object." (Peirce, CP 1.564). In contrast to the traditional dyadic models, Peirce defines a sign as a triad. And this implies that a sign does not stand for its object in all respects, "but in reference to a sort of idea, which I have sometimes called the ground of the representamen. 'Idea' is here to be understood in a sort of Platonic sense, very familiar in everyday talk" (Peirce, CP 2.228 and 4.536)).

This implies that the sign is consciously recognized by the cognitive subject and for that purpose the subject has to create another sign, which becomes an interpretation of the first interpretant. As Roman Jakobson, characterizing Peirce's thinking, once said:

"One of the most felicitous, brilliant ideas which general linguistics and semiotics gained from the American thinker is his definition of meanings *as the translation of one sign into another system of signs* (4.127)" (Jakobson 1985, 251).

The flow of meaning thus expresses the contradiction and it evolves by a recursive interaction between the objects (referents) and interpretants (senses) of signs. Objects and interpretants of signs are in general signs themselves. We argued elsewhere (Otte, 2003) in great detail that (mathematical) meaning has two components, one of which refers to objects, and which is called the extensional component of meaning; the other relating to the interpretant of the sign and which it is suitable to call the intensional or coherence component. The most important consequence, to be applied in the following paragraphs, consists in the fact that there never is a definite meaning; neither in the structural or intensional sense nor with respect to the extensions of theoretical terms. A pragmatic perspective on things thus seems to always recommend itself.

All reasoning is an interpretation of signs of some kind. And the interpretation of a sign is nothing but the construction of a new sign. As was said above, a mere feeling or consciousness, without a representation, is no interpretation and an interpretation or reformulation of a text, which does not carry on the ideas and does not generalize, is futile also. All cognition proceeds by means of the construction of an adequate representation and this construction provides nothing but the contradiction between subject and object with a form. "It is a contradiction that a body will permanently fall into another and at the same time will flee away from it. The ellipse is a form of development by which this contradiction is as much realized as it is resolved" (K. Marx, see above).

A symbol mediates between subjective spontaneity and objective reaction and is termed a Third, by Peirce.

The object of knowledge, being nothing but a representation—something which Kant had dubiously called an intuition—therefore is also not something given "out" there, it is not a Kantian "thing in itself," but is established by the relation between subject and reality. It makes itself felt equally by the objectivity of this interaction process as well as through its breaking downs.

Mathematical ontology, for example, is constituted by a practice of mathematical reasoning and application, not the other way around. A mathematical object, such as number or function, does not exist independently of the totality of its possible representations, but must not be confused with any particular representation, either. We have on a different occasion expressed these facts in terms of a principle of complementarity (Otte, 2003). To see how a semiotic perspective might help to better grasp that complementarity one should remind oneself of the following characteristics of mathematics;

- Mathematics, on the one hand, has no more concrete objects of its own than painting; it is therefore not possible to do mathematics by simply considering certain kinds of objects, either constructed or given, abstracting what seems essential

about them. According to the Cantorian claim that consistency is sufficient for mathematical existence, there is so much truth that it is consistency which makes a sign potentially meaningful. Consciousness "is sometimes used to signify the (Kantian) *I think,* or unity in thought; but unity is nothing but consistency, or the recognition of it. Consistency belongs to every sign, so far as it is a sign; therefore every sign, since it signifies primarily that it is a sign, signifies its own consistency" (Peirce, CP 5.313-15).

- On the other hand, mathematics is not a mere logical language, nor is it an analytical science from concepts, that is, definitions. Mathematics includes indexical representations and observational activities. "The best thinking, especially on mathematical subjects, is done by experimenting in the imagination upon a diagram or other scheme," says Peirce (Peirce, NEM I, 122).

Thus the idea of a sign might help us to better understand that these different characterizations of mathematics are not as distinct as it might have appeared at first sight, but rather they represent complementary aspects of mathematical thinking, because signs are always used referentially as well as attributively. This is but another expression of the interaction between object and interpretant of the sign, as indicated above.

The semiotic approach to cognition and epistemology distinguishes itself from the philosophy of consciousness (as developed by Kant, for example) by its radical break with the assumptions and prerequisites of reasoning characterizing the latter. "All our thinking," says Peirce, "is performed upon signs … External signs answer any purpose, and there is no need at all of considering what passes in one's mind" (Peirce, NEM I, 122). Thinking occurs in signs and representations, rather than by means of imaginations or intuitions, which are to be looked for within our heads. This does not mean that conscious recognition and intuitive activity are dispensable. It only means that they have to be taken as means and instruments of cognitive activity, rather than as its foundations (Otte, 2005, 16f).

Insisting, when for example trying to interpret a text, on the question "what did the author really mean" has no more merits to it than the idea that the reader, and not the author, is the sole source of meaning. "Not even the author can reproduce his original meaning because nothing can bring back his original meaning experience" (Hirsch, 1967, 16; and in contrast: Fish 1980, 359f). And correspondingly, not any arbitrary reformulation of a text is an admissible interpretation. Neither the author nor the reader is the unique source of meaning because meaning is but the sign process itself. The reality of a text is its development, the meaning of a proposition lies in its consequences and the essence of a thing is the essence or meaning of a representation of that thing, and so forth. The semiotic approach fosters a genetic perspective on knowledge. Knowledge is essentially a process, a learning process or a process of growth and generalization, expressed in terms of a permanent transformation of one representation into another one.

Imagining cognition as a contradiction between subject and object implies the conviction that neither subject nor object can dominate or even determine the other part of this relationship. We do not find final and definite descriptions of things and mostly we do not even know what we know. We apply it, we represent it, but we

cannot say or express it, nor describe what we are doing. “What can be shown cannot be said,” Wittgenstein famously affirmed. The spirit of creative activity thus is more or less the following.

Everything that we have formulated or constructed is just done and is there in the plain light of day. It means nothing *per se*, it is just there. Everything we achieve, we simply achieve. It neither needs nor deserves an interpretation or commentary, because it is, as we perceive it, real. The commentary would add nothing to the thing created and given. The given is just the given. What we have made, we have made. It has no general symbolic significance nor can it be undone. An action is an action, a work of art is just a work of art, a theory is just a theory. It must be grasped as a form *sui generis,* and recreated in its own terms, before we can inquire into its possible meanings or applications. Any creative achievement remains imperfect as long as questions about its meaning dominate when considering it. In artistic drawing what we achieve is a line, and the line does all the work, and if it fails to do so no philosophical commentary will rescue or repair a bad work of art. In literature or philosophy, it is the word or the sentence, in mathematics the new concept or the diagram, which carry the entire weight, etc. etc. Mastery, Paul Valery, says, presupposes that “one has the habit of thinking and combining directly from the means, of imagining a work only within the limits of the means at hand, and never approaching a work from a topic or an imagined effect that is not linked to the means” (Valery, 40).

Everything just is and thus means itself: P=P! This principle of identity lies at the heart of art and likewise at that of logic or exact science and it is obviously directed against any idea of cognition as a mental feeling or inner experience. P just means P! No commentary and no psychological experience or philosophical consideration shall be able to add anything to the matter.

A monotonous and perfect repetition would, however, destroy any creativity as well. Any line in an artistic drawing is, in fact, a continuum of lines; it fulfills its destination to represent something, at the very same time indicating an indeterminate set of possible modifications and further developments.

The creative process thus operates on the interplay of variation and repetition. A theory or a work of art, being an interpretation, is also a process, namely the process of creating an interpretant of the representation given and so on. At this very moment we are developing the anti-thesis, that is, pointing to the fact that a work of art or a theory are not mere existents, but are signs, which have a meaning. And an interpretation of that meaning is nothing but another representation. The sign is thus a thing as well as a process, namely the process of establishing a relationship between object and interpretant. It is a flow of meaningfulness. Peirce, in fact, defines semiosis as the action or process of a sign. “By ‘semiosis’ I mean”, Peirce writes, “an action, or influence, which is, or involves, a cooperation of three subjects, such as a sign, its object, and its interpretant, this tri-relative influence not being in any way resolvable into actions between pairs” (Peirce, CP 5.484).

Evolutionary realism therefore means the co-evolution of reality and knowledge, that is, the evolution of symbolism. It is the symbol in movement.

II. Let us now try and spell out the problem to which we should like to apply our semiotic view of mathematical activity. This

is in fact the problem of mathematical explanation.

There has been, for some time now, a widespread debate about mathematical explanation and rigorous proof in mathematics education as well as in the philosophy of mathematics (for an overview see Mancosu, 2000 and 2001; Hanna, 2000). In this discussion, a distinction between proofs that prove against proofs that explain has over and again played an important part. Gila Hanna, for example, presents the distinction in psychological terms, but later on describes *explaining* in this way: "I prefer to use the term explain only when the proof reveals and makes use of the mathematical ideas which motivate it. Following Steiner (1978), I will say that a proof explains when it shows what 'characteristic property' entails the theorem it purports to prove" (Hanna 1989, 47).

Hanna and Steiner, speaking about the "characteristic property" that entails the "theorem it purports to prove," seem to follow Bolzano respectively as well as Aristotle in their ideas about mathematics. The "characteristic property" seems something like an essential cause in the Aristotelian sense. Steiner's view "exploits the idea that to explain the behavior of an entity, one deduces the behavior from the essence or nature of the entity" (Steiner 1978, 143). Steiner, believing that all mathematical truths are necessary and are thus valid in "all possible worlds," prefers to use the term "characterizing properties," rather than the term "essence." But he makes very clear his belief that mathematical proofs are exclusive like calculations or numerical determinations, picking out "one from a family" (147), rather than being general proof schemes or general forms of argumentation and demonstration. This view appears to be derived from an Aristotelian model of science and mathematics and it stands in extreme contrast to modern axiomatical mathematics in the sense of Hilbert or Emmy Noether, for example.

The proofs of modern mathematics are not glued to the particularities of individual propositions and it is generality of perspective and fertility of method that render them explanatory, because it is this which opens up new possibilities for mathematics. A proof is first of all a sign or representation and, as such, is a general already. It is the objectivity of general relationships what matters. Even if one were concerned with the subjective or educational aspects of the matter and therefore interested in the intuitive insights of a proof, this would primarily imply, as we have indicated in Part I, the search for new applications or representations of the basic ideas.

The distinction Steiner and others have drawn between proofs that explain and proofs that merely prove or verify makes sense only with respect to an Aristotelian model of science, as it is exemplified, for instance, by Euclid's *Elements* of geometry. This Aristotelian model has been described by E. Beth (1968) and more recently by de Jong (2003). An Aristotelian science, according to these descriptions, is comprised of a system of fundamental concepts such that any other concept is composed and is definable in terms of these fundamental concepts; it also contains a system of fundamental propositions such that all other propositions are grounded in and are provable from these fundamental propositions. And the fundamental concepts or propositions stand in close continuity with everyday thinking. Explanation in such a context means reduction to the concrete foundations of general experience, rather than constructing

new theoretical contexts and searching for new applications.

Bolzano, in fact, referring to Aristotle, seems to have been the first modern author pleading for demonstrations «that show the objective connection and serve not just subjective conviction.» His monumental "Wissenschaftslehre" (doctrine of science; 1836/1929) was conceived of as a general science or logic in the service of enlightenment and was organized like a didactical treatise. This work contains a distinction between proofs that merely prove, being intended to create conviction or certainty, and others, which "derive the truth to be demonstrated from its objective grounds. Proofs of this kind could be called justifications (Begruendungen) in difference to the others which merely aim at conviction (Gewissheit)" (Bolzano, Wissenschaftslehre, vol. IV, p.525, 261). In an annotation to this paragraph Bolzano mentions that the origin of the distinction goes back to Aristotle and the Scholastics, who have, however, attributed an exaggerated importance to it by affirming that only justifications produce genuine knowledge, but that it had fallen into neglect in more recent times.

On grounds of this distinction between proofs that are merely certain and others which are genuine justifications, Bolzano criticized Gauss' proof of the fundamental theorem of algebra of 1799, for example, because Gauss had on that occasion employed geometrical considerations to prove an algebraic theorem. Bolzano did not, as is often claimed (Volkert 1986), doubt the validity of Gauss' arguments and he did not question the certainty of our geometrical knowledge, but criticized the "impurity" of Gauss proof.

It is this spirit that led to the so-called rigour movement and to the program of arithmetization of mathematics and Bolzano has in fact been one of the spiritual fathers of this program. Mathematics was to be established as an analytical science from definitions, and numbers were considered to be the most important means of mathematical analysis.

One important effect of this program was the separation between pure and applied mathematics and the reconstruction of pure mathematics on completely logical, or rather, conceptual terms. Continuous mathematics, like geometry, for example, was considered applied mathematics. All intuitions and objects were to be replaced by definitions and mathematical proof, becoming the central concern of mathematicians, should be performed as a kind of linguistic activity. Although the conceptions of logic involved varied considerably, mathematical explanations in the end amounted to nothing but rigorous deduction from first principles and basic concepts.

One of Bolzano's most important mathematical achievements was the proof of the existence of the least upper bound of a bounded set of real numbers and, based on this, a completely analytical proof of the intermediate value theorem for continuous real functions. Both results were published in 1817 in Bolzano's "Rein analytischer Beweis des Lehrsatzes, dass zwischen zwei Werten, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege." Bolzano's proof of the intermediate value theorem survives nearly unchanged in today's calculus textbooks, although one aspect has changed fundamentally since Dedekind. Bolzano had based his proof on the Archimedean axiom, which says that given any two real numbers A and B, there will always be a natural number n such that

n*A* supersedes *B.* He had, however, taken this axiom to be an obvious truth, rather than a postulate. It was Dedekind only, who realized that nothing of such a kind could be proved or assumed as obvious. As Dedekind states it with respect to his own definition of continuity:

"The assumption of this property is nothing else than an axiom by which we attribute continuity to the line, by which we think continuity into the line. If space has real existence at all it is not necessary for it to be continuous" (Dedekind 1912, p.3, my translation).

The filling-up of gaps in the rational numbers through the creation of new point-individuals is the key idea underlying Dedekind's construction of the domain of real numbers. Bolzano, in contrast, thought it obvious that these points, as exemplified by the incommensurability of certain line segments, for example, existed objectively. Charles Sanders Peirce's view of the continuum is, in a sense, intermediate between that of Dedekind and Bolzano. He held that the cohesiveness of the continuum rules out the possibility of it being a mere collection of discrete individuals, or points, in the usual sense. "A *continuum* is precisely that every part of which has parts, in the same sense" (Peirce, W2, 256). The continuum represents the reality of the possible determination of points, rather than be an actual set of points; but this possibility is objective, such that, differently from Dedekind, space could not be discrete, according to Peirce.

If one looks at the various proofs of the intermediate value theorem one might be inclined to ask: why not take this theorem itself as the essential continuity postulate? It seems as clear and obvious as any of the other candidates, the existence of the limit of a bounded monotonous sequence, the Heine-Borel theorem, the existence of a point of intersection of a nested sequence of closed intervals of rational numbers with lengths tending to zero, etc. etc.

Mainly pragmatic reasons are responsible for the choice of axioms, reasons that are related to the development of mathematical knowledge and the construction of theories. But what about the problem of explanation then? To explain amounts to exhibiting the meaning of something. Mathematics has, however, no definite meanings, neither in the structural intra-theoretical sense nor with respect to intuitive objectivity. Signs and meanings are processes, as we have argued in paragraph I.

Resnik and Kushner do not consider the proof of the intermediate value theorem as explanatory in the sense of Steiner's characterization. They write:

"We find it hard to see how someone could understand this proof and yet ask *why* the theorem is true (or what makes it true). The proof not only demonstrates how each element of the theorem is necessary to the validity of the proof but also what role each feature of the function and the interval play in making the theorem true. Moreover, it is easy to see that the theorem fails to hold if we drop any of its conditions" (Resnik/ Kushner 1987, 149).

Rigorous proofs in this sense do not admit "why"-questions any more than mere calculations do and it is hard to see how they could be explanatory at all. Considering the question of how to choose the relevant mathematical model might perhaps change the situation. But the reader should remind herself that the term "explanation" had, for Bolzano, an

objective meaning, rather than a psychological one. And this objectivism led to his error with respect to the foundations of the real numbers and his ignorance of the fact that mathematics contains only hypthetico-conditional statements, rather than categorical ones. This, however, means that the foundations of mathematical claims lie, so to speak, "in the future", in the use and application of the mathematical propositions. A mathematical proof must therefore generalize in order to be explanatory. As we have seen, however, with respect to Bolzano and Steiner or Hanna, there is a strong foundational tendency involved in their ideas of explanatory proofs. It is very essential to Bolzano, for example, that there exist a hierarchy of truths in themselves independent from our knowledge or representation.

Cauchy had, at about the same time as Bolzano, given a geometric argument for the intermediate value theorem, being more concerned with certainty and conviction than with objective foundation (Cauchy 1821, 43f). Bolzano did consider proofs, like those by Gauss or Cauchy, as sufficiently obvious and convincing, but objected that they did not show the real fundamentals and thus were not true justifications, but rather mere subjective confirmations (subjektive Gewissmachungen). It is clear, Bolzano writes, "that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e. arithmetic, algebra analysis) from considerations that belong to a merely applied or special part, namely geometry. … For in fact, if one considers that the proofs of the science should not merely be convincing arguments, but rather justifications, i.e. presentations of the objective reason for the truth concerned, then it is self-evident that the strictly scientific proof, or the objective reason of a truth which holds equally for all quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in space. On this view it may on the contrary be seen that such a geometrical proof is really circular. For while the geometrical truth to which we refer here is extremely evident, and therefore needs no proof in the sense of confirmation, it nonetheless needs justification" (Bolzano after the translation by Russ 1980, 160).

The term "justification" refers to the Leibnizian idea that every concept can be decomposed into "atoms." Unprovable or basic propositions must, according to Bolzano, be among those whose subjects and predicates are completely simple concepts in the sense of Leibniz. Bolzano believed, for example, that different cases of one and the same issue should be formulated in terms of different propositions, like in Euclidean geometry. The law of cosine, for instance, in the cases of the acute- respectively obtuse-angled triangles signifies two different cases requiring different arguments. "Euclid was right in formulating two different propositions here," writes Bolzano (Bolzano 1810/1926, 61).

Bolzano not only emphasized the necessity of "homogeneity" between method and object but he also conceived of concepts in themselves, propositions in themselves and representations (Vorstellungen) in themselves, independent of our thinking about them. This is sometimes emphasized by saying that Bolzano was the first to realize that "the proper prolegomena to any future metaphysics was the study of what we say and its laws and that consequently the prima philosophia was not metaphysics or ontology but semantics" (Bar-Hillel, 1967,

337f). Thus Bolzano's objective semantics and the Platonic and hierarchically structured universe of objective meanings is essential to his whole conception of explanation.

There are close parallels between Peirce and Bolzano and they are due to the fact that both their philosophies resemble that of Leibniz very strongly indeed. Both did, however, modify classical ontologism, concentrating on how mathematicians create and communicate as well as on the semantics of mathematical affirmations or communications. Both also consider mathematics as the science of possibility or of the possible states of affairs and both understand that proofs do not exist independently from mathematical theories, but are parts of theories.

Finally, both Bolzano and Peirce were concerned with elaborating alternatives to the philosophy of consciousness, as exemplified by Kant's *Critique* and his notion of *a priori* intuition in particular; however, Bolzano denied the evolutionary perspective, saying that Kant had confounded mathematics as such with the way in which humans develop mathematics, whereas Peirce, in contrast, sought to provide evolutionism with an objective basis. The continuity of space and time is objective, rather than subjective, as Kant and Leibniz had believed.

The essential difference between Bolzano and Peirce lies in the way how possibility is conceived. Bolzano thinks about this question in terms of the difference between the actual and the possible. This means that something like the set of all possibilities exists a priori, waiting to possibly be actualized. For Peirce, in contrast, reality is an evolutionary process realizing and producing objective possibilities as well as their conditions.

Peirce over and again stressed that we have to explain not only phenomena but also the laws that govern them (Peirce W4, 551f, see also Peirce, CP 1.175). Peirce, unlike Bolzano, did not consider mathematics to be an analytical science from definitions. Reality is continuous and thus cannot be described or determined. This may even be interpreted on the level of mathematics. Peirce in contrast to Bolzano seems well aware of the fact that there may exist various models of the number line.

The main feature of mathematical reasoning lies therefore in its perceptual character and consists in the fact that all "deep" symbolic meanings must have been eliminated, in the same sense we have described creative activity in Part I above. A proof must enlarge our knowledge and all ampliative or synthetic reasoning is perceptual and inductive, or as Peirce sometimes calls it, "abductive." This does not contradict the fact that mathematical reasoning is necessary, because "no necessary conclusion is any more apodictic than inductive reasoning becomes from the moment when experimentation can be multiplied *ad libitum* at no more costs than a summons before the imagination" (Peirce, CP 4.531). Hence, it amounts to the same to say that mathematics "busies itself in drawing necessary conclusions," and to say that it occupies itself with ideal or hypothetical states of things (Peirce, CP 3.558).

Mathematical proofs in the sense of Peirce do not contain explanations. They consist of apodictic judgments, showing clearly that something is necessarily the case, rather than explaining why that something is the case. They are examples of "knowledge that," rather than "knowledge why" in the sense of the Aristotelian

distinction between proofs of the fact (hoti) and proofs of the reasoned fact (dioti). “The philosophers are fond of boasting of the pure conceptual character of their reasoning. The more conceptual it is the nearer it approaches to verbiage” (Peirce, CP 5.147-489). This would sound Kantian, were it not for the reference to the importance of signs.

Already from the fact that a proof is a sign and a sign is determined by its object and combined with the requirement that mathematical proofs are necessary and thus apodictic, it follows that a proof is essentially an icon and that its object is nothing but the form of that icon. Peirce affirms that mathematical reasoning proceeds by means of the construction of all kinds of diagrams and by experimenting with them and observing what happens. “Since a diagram .... is in the main an Icon of the forms of relations in the constitution of its Object, the appropriateness of it for the representation of necessary inference is easily seen” (Peirce, CP 4.531).

Peirce took Leibniz’s theory of a continuum of representations from quite unconscious and quasi imperceptible representations to those most coercive to consciousness and subsequently based his whole semiotic epistemology on it. A realistic view must see reality above and beyond all laws, ideas and explanations as something offering the possibility of understanding. Peirce’s metaphor for such a view of reality is the continuum. Reality is commonly identified with the totality of existing objects and facts. Sometimes, in a flush of enlightened insight, relations or laws are added to the furniture of reality. But this does not help much. The set of all laws, or possibilities of things, is a no less an antinomical conception than the notion of the set of all sets, which lies at the bottom of Russell’s paradox. In a digital or discrete world, with only 1 and 0, or perfectly right and wrong, there would be no growth of knowledge and therefore no knowledge at all. Synechism is above all “a regulative principle of logic prescribing what sort of hypothesis is fit to be entertained and explained” (Peirce, CP 6.173). Or, stated somewhat differently, only a continuous reality makes analysis and inductive generalization possible. According to Peirce relations are not to be reduced to determinate relata, but are related to continua. This was as important to the geometrical illustrations of the classical incommensurability proofs as it was important to the foundations of the calculus. Leibniz had already emphasized these epistemological insights, but had remained bound to a substance ontology in the Aristotelian sense.

What primarily characterizes mathematics is the peculiarity of its generalizations by means of the forming of fertile hypotheses. A “hypothesis substitutes, for a complicated tangle of predicates attached to one subject, a single conception” (Peirce, W3 337). Such hypotheses are created by an inductive process which Peirce called abduction or abductive inference, adding that “abductive inference shades into perceptual judgment without any sharp line of demarcation between them” (Peirce, CP 5.181). Abductive reasoning involves an element of intuition and “intuition is the regarding of the abstract in a concrete form, by the realistic hypostatization of relations; that is the one sole method of valuable thought” (Peirce, CP 1.383). This realistic hypostatization occurs by means of the construction and experimentation with all kinds of diagrams. From the abductive suggestion, which synthesizes a multitude of predicates, «deduction can draw a prediction» (Peirce, CP 5.171).

Thus the meaning and foundations of a piece of mathematical knowledge, a theory, for instance, are to be seen in the intended applications and newly created possibilities. Icons or images are particularly well suited to make graspable and conceivable the possible and potential rather than the actual and factual. It should also be mentioned in this context that psychology and psychotherapy have known for some time that icons or images are particularly well suited to strengthening what could be called "sense of possibility" and which seems indispensable to a person's mental health (see the proceedings of the 35th International Congress on Psychoanalysis in San Francisco, 1995). Confining a person—a student, for example—to a certain characterization of herself/himself would mutilate her/his personality. Mathematical explanation must therefore enlarge and widen the perspective of the addressee of the explanation and the real is generally to be conceived of as process and evolution.

III. It is rather common nowadays to contrast subjective insight and explanation with objective foundation and conviction (Hersh, 1993). Indeed, Hanna's quest for insight and understanding seems completely psychological and has nothing to do with objective concerns. Bolzano, in contrast, maintaining a strong anti-psychologistic attitude, conceives of explanation in purely objective or logical terms and in reference to a world of truths in themselves, independent of any actual insight. When in the course of the 19th/20th centuries the humanities (Geisteswissenschaften) were developed by W. Dilthey (1833-1911) and others, it became common to contrast understanding and interpretation, as the basis of the humanities, with scientific and mathematical explanation. This distinction resulted later on in the notion of the "two cultures" (Snow). Snow's basic thesis was that the breakdown of communication between the sciences and the humanities (the «two cultures» of the title) was a major hindrance to solving the world's problems (see C.P. Snow, 1993)

How can both sides come together? We believe that these two different views can be reconciled from a genetical perspective and that for this the semiotic view and the idea of mathematics as mathematization are essential. The notion of interpretation should be transformed as outlined in Part I of this paper and scientists and mathematicians should refrain from the metaphysical realism and logical objectivism that tends to identify reality with our knowledge of it, thus confusing object and sign.

A mathematical proof is a type, a type of representation, rather than a token-construction. One has to grasp the integrated whole of it, not merely follow the argument or the calculation. Or rather, one has little choice here, as one will hardly be able to memorize a complex proceeding and repeat its application without analysis and generalization.

Still this does not commit us to Platonism, as an idea is not completely to be dissociated from its possible applications and the applications might affect our conviction about what is essential or basic. And to understand the logic of an argument, one must not only follow its consequences in the abstract, but must also see how it applies in a particular situation. Resnik and Kushner found it hard, as they wrote, to see how someone could understand the proof of the intermediate value theorem "and yet ask *why* the theorem is true (or what makes it true)." They are right. This kind of

insistence on more and more new why-questions seems to happen when one separates knowledge from its development and application. But the meaning resides in the applications.

In formal mathematics, facts are explained by means of proofs and then it has to be proved that the proof is correct and so on *ad infinitum*. Every proof is faced with the prerequisite of proving that the proof be correct. And the proof of the correctness of the proof again meets the same requirement and the proof of the correctness of the correctness of the proof also ... etc. This dilemma is nicely described by Lewis Carroll's version of Zenon's paradox (Carroll, 1905*;* see also: Peirce, CP 2.27).

As a rational being one cannot act contrary to one's own insights and there is no insight without an application. Lewis Carroll's version of the race between Achilles and the Tortoise shows, albeit unintentionally, that one cannot really have knowledge or an insight and keep from applying it. There is no complete analysis without activity and application. Mathematics is just as constructive as it is analytical. Hence, it is difficult to believe that mathematics is meant "to explain," in the usual reductionistic understanding of the term.

In a reader on the philosophy of science we are told: "We can explain the length of the shadow by reference to the height of the flagpole, and not vice versa" (Newton-Smith 2000, 129). It seems natural to ask, upon perceiving a shadow, whence it comes from. Nobody, however, would consider the shadow to be the cause of the flagpole. But what about mathematics? Let us begin with Kant.

A "new light," says Kant, must have flashed on the mind of people like Thales, when they perceived that the relation between the length of a flagpole and the length of its shadow enables one to calculate the height of the pyramid, given the length of its shadow. "For he found that it was not sufficient to meditate on the figure as it lay before his eyes,... and thus endeavor to get at knowledge of its properties, but that it was necessary to produce these properties, as it were, by a positive a priori construction" (Kant, *Critique of Pure Reason*, Preface to the Second Edition 1787). And indeed, the flagpole as such has no positive relationship whatsoever to the pyramid.

Now one might say that mathematics is not concerned with flagpoles, pyramids and the like. But such talk does not help very much, given that we have witnessed, since Descartes' arithmetization of geometry, a gradual destruction of the pre-established harmony between method and object of mathematical inquiry that Bolzano wanted to maintain (Boutroux 1920, 193f). The history of mathematics must be seen as the history of mathematization, including the mathematization of mathematics itself (Lenhard y Otte, 2005). Therefore, mathematics is characterized first of all by the manner in which it generalizes. Mathematicians as a rule do not see things this way. They are either Platonists or Intuitionists and they dislike the semiotic approach to mathematics (Hermann Weyl is a noticeable exception to this: see: *Werke*, vol. IV, p. 334).

G. Cantor (Cantor 1966, 83), for example, believed that applied mathematics must deal with real explanations or foundations of things and thus must be based on sound metaphysics, whereas pure mathematics

is defined by its "freedom" to form concepts as one pleases (given that they do not result in logical contradictions). Kant, on the other hand, being confined to an epistemology of consciousness, found it necessary to employ the idea that mathematical concepts and relations must be "constructed in intuition." And people like Poincare or Brouwer followed him in this conviction. This, however, imposes severe limitations on the conception of mathematics, because it introduces an absolute distinction between concepts and intuitions and between analytical and synthetical knowledge.

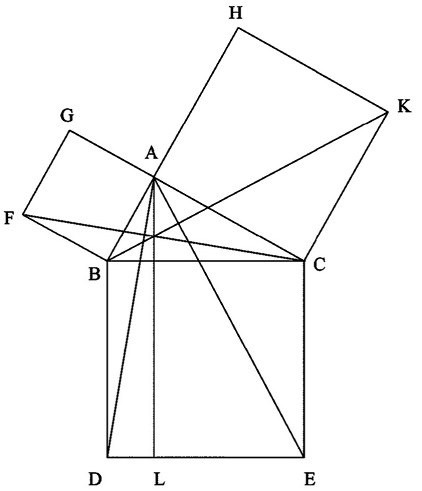
Peirce considered these distinctions as relative and hence his belief that abduction, as the source of mathematical generalization, on the one hand, and empirical perception, on the other hand, are not as different as it may appear. In semiotics, to explain means to represent. And a representation is just a perception cast into a certain form. In this context, Peirce develops the notion of the perceptual judgment as an unconscious inference. There is no sharp demarcation between mathematical and perceptual judgments respectively. When making a perceptual judgment we simply cannot really distinguish between what comes from the outside world and what stems from our own interpretation. "On its side, the perceptive judgment is the result of a process, although of a process not sufficiently conscious to be controlled, or, to state it more truly, not controllable and therefore not fully conscious. If we were to subject this subconscious process to logical analysis … this analysis would be precisely analogous to that which the sophism of Achilles and the Tortoise applies to the chase of the Tortoise by Achilles, and it would fail to represent the real process for the same reason" (Peirce, CP 5.181).

Within a perceptual judgment, the perception of generals (or ideal objects) and of particular data seems inseparable, or, stated differently, the processes of creation and of application of symbolic representations are inseparable. Analysis and interpretation interact. The relativity of the distinction between our inner and outer world could thus be interpreted as demanding its conceptualization in interactive terms, like the concept of representation. Once more we have to conclude that a proof that is supposed to explain must generalize.

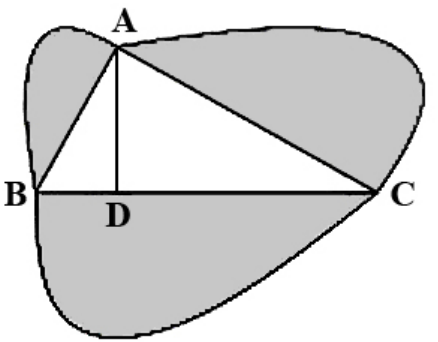
Let us consider a concrete example, given by Boulignand (1933), which concerns three different proofs of the Theorem of Pythagoras. The proofs of the Pythagorean Theorem are commonly considered to be divided into three main types: proofs by *shearing*, which depend on theorems that the areas of parallelograms (or triangles) on equal bases with equal heights are equal, proofs by *similarity* and proofs by *dissection*, which depend on the observation that the acute angles of a right triangle are complementary. Among these proofs the proofs by similarity play a special role because they indicate their embeddedness into the theoretical structure of axiomatized Euclidean geometry. The Pythagorean Theorem is equivalent to the Parallel Postulate, after all.

The following diagrams represent examples of these three types of proofs.

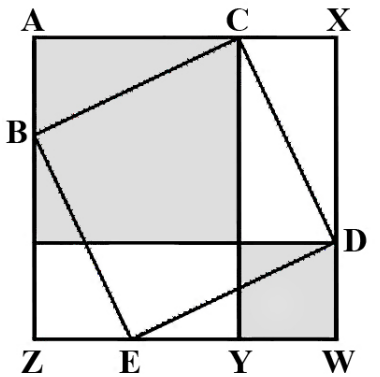
1.



2.



3.



The first proves, the second explains and the third is called intuitive but not explanatory by Boulignand.

The first proof proceeds in the traditional manner that we have become accustomed to in school: Since the angles BAC and BAG are right it follows … Consider now the triangles ABD and FBC … Since the triangles are congruent it follows that …. etc.etc.…

The second proof requires a relational understanding of the notion of "area," rather than an empiricist one. The area of a figure is defined then as the relation of that figure to the unit square $Q(1)$. We have $Q(x)=x^2\,Q(1)$. Therefore the areas of similar plane figures are to each other as the squares of their corresponding sides. Since we have ADC+ADB=ABC, the generalized theorem of Pythagoras follows.

The third proof simply requires some playing around with plane figures like in a geometrical puzzle and observing certain concrete relationships of equality and difference.

The interesting distinction seems to be that between 2) and 3), whereas the distinction between 1) and 2) is familiar and in some way refers to the well-known distinction between the analytic and synthetic, or between corollarial and theorematic reasoning. Corollarial reasoning relies only on that which is enunciated in the premises in a rather straightforward manner. If, however, a proof is possible only by reference to other things not mentioned in the original statement and to be introduced by conceptual construction and generalization, such a proof is theorematic.

The first idea that comes to mind with respect to the contrast between 2) and 3) is that it must be something modern,

because it has to do with relational thinking and with the opposition between theoretical thought and common knowledge, or between the exact sciences and the humanities (Dilthey). We have talked about this difference already and one should remember the fact that Euclidean axiomatics and modern axiomatics in the sense of Hilbert are representing this difference (Otte 2003, 204). What is more important still: in modern axiomatic theory mathematical objects or facts are the objects and facts of a theory and proofs only make sense within the context of a theory? In traditional Euclidean geometry all this is different. The objects are given by unaided intuition, independently of any theory, and the proofs do not refer to an explicit and fixed theoretical context as their base, but refer to everyday rationality in the sense of Aristotelian demonstrative science.

Now, the second proof is modern in the described sense, whereas the other two more or less breathe in the spirit of Aristotelian science and traditional thinking in terms of substances and their essential properties.

When classifying the second proof as explanatory, we employ a dynamic conception of knowledge and explanation, as it has been described in semiotic terms above. The proof indicates the possibility of many relationships and thus makes us feel the systemic and theoretical character of knowledge. The other two proofs are foundationalist, assuming a fixed hierarchical organization of knowledge based on unaided intuition and everyday experience.

Intuition seems forceful, but neither an absolute insight or intuition nor a determinate hierarchy of levels of knowledge actually exist. This is very often misunderstood. For example, the well-known Gestalt psychologist Max Wertheimer (1880-1943) comments on the presentation and solution of Zeno's paradoxes by means of a geometric series that is current in present day mathematics. He himself comments on the current proof of the convergence of that series, which is accomplished by multiplying the series by a and subtracting afterwards. Set $S = 1 + a = a^2 + ...$ Then $S - aS = 1$ or $S = 1/(1 - a)$.

Wertheimer writes: "It is correctly derived, proved, and elegant in its brevity. A way to get real insight into the matter, sensibly to derive the formula is not nearly so easy; it involves difficult steps and many more. While compelled to agree to the correctness of the above proceeding, there are many who feel dissatisfied, tricked. The multiplication of $(1 + a + a^2 + a^3 + ...)$ by a together with the subtraction of one series from the other, gives the result; it does not give understanding of how the continuing series approaches this value in its growth." (Wertheimer, 1945)

Wertheimer wants an intuitive demonstration. Intuition is essentially the seeing of the essence of a thought or object as a form or object itself. Things do not have, however, a unique and demonstrable essence, as we have argued before. The essence of something cannot be anything but the essence of a representation of that thing and therefore the diagrammatic proof which Wertheimer does not accept as satisfactory, could be called an intuitive proof, exactly like proof number 3 of the theorem of Pythagoras above. Only, in the present case, the intuition is directed towards the diagrammatic representation itself and to its form. It is also more advanced, because it contains some general methodological message.

If we could establish a direct authentic and "natural" relationship to the object of knowledge then this relationship would also exist in a mechanical form; it would be a relation between reactive systems rather than cognitive ones and thus would be just a singular event without general meaning. The idea of sign marks the difference at this point as it introduces a general element. Our intuitions serve to create expressive and illuminating representations. And in this way we learn to act within the world around us. To understand means exactly to create a representation, as the very example that Wertheimer has criticized shows. We therefore have to renounce searching for definite meanings and absolute foundations of knowledge.

This we can learn from the fact that all our thinking is by means of signs.

Classified in terms of Peirce's categories, the third or intuitive proof represents *Firstness*, the first *Secondness* and the second, or explanatory in our sense, *Thirdness*. Thirdness is, as Peirce says, a synonym of representation and evolution and thus of continuity (CP 6.202). But Thirdness presupposes Firstness and Secondness, or stated semiotically, symbolic representation depends on iconic and indexical elements. Thus a proof may be a symbol, but mathematical reasoning is, as was said, diagrammatic and as such is based mainly on iconic signs with indexical elements as parts of the icon. As Peirce adds: "Firstness, or chance, and Secondness, or brute reaction, are other elements, without the independence of which Thirdness would not have anything upon which to operate" (CP 6.202). What primarily characterizes mathematics is the peculiarity of its generalizations and this is a symbolic process operating by means of hypostatic abstractions (Otte 2003, 218f).

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