

# Semiotic Objectifications of the Compensation Strategy: En Route to the Reification of Integers

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## RESUMEN

Reportamos aquí el análisis de una experiencia que reproduce el trabajo de investigación “Object-Process Linking and Embedding” (OPLE) en el caso de la enseñanza de la aritmética de los enteros, desarrollada por Linchevski y Williams (1999) en la tradición de la Educación Matemática Realista (realistic mathematics education (RME)). Nuestro análisis aplica la teoría de la objetivación de Radford, con el propósito de aportar nuevas pistas sobre la forma en que la reificación tiene lugar. En particular, el método de análisis muestra cómo la generalización factual de la estrategia llamada de compensación encapsula la noción de “agregar de un lado es lo mismo que quitar del otro lado”; una base fundamental de esto que será, más tarde, las operaciones con enteros. Discutimos, de igual modo, otros aspectos de la objetivación susceptibles de llegar a ser importantes en la cadena semiótica que los alumnos ejecutan en la secuencia OPLE, secuencia que puede llevar a un fundamento intuitivo de las operaciones con los enteros. Sostenemos que es necesario elaborar teorías semióticas para comprender el papel vital de los modelos y de la modelación en la implementación de las reificaciones en el seno de la Educación Matemática Realista (RME).

● **PALABRAS CLAVE:** Enteros, semiótica, teorías del aprendizaje.

## ABSTRACT

We report an analysis of data from an experimental replication of “Object-Process Linking and Embedding” (OPLE) in the case of integer arithmetic instruction originally developed by Linchevski and Williams (1999) in the realistic mathematics education (RME) tradition. Our analysis applies Radford’s theory of semiotic objectification to reveal new insights into how reification is achieved. In particular the method of analysis shows how the factual generalization of the so-called compensation strategy encapsulates the notion that “adding to one side is the same as subtracting from the other side”: a vital grounding for symbolic integer operations later. Other aspects of objectification are discussed that are considered likely to be important to the semiotic chaining that students achieve in the OPLE sequence that can lead to an intuitive grounding of integer operations. We

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argue that semiotic theory needs to be elaborated to understand the vital role of models and modelling in leveraging reifications in RME.

● **KEY WORDS:** Integers, Semiotics, Theories of Learning.



## RESUMO

Reportamos aqui o análise de uma experiencia que reproduce o trabalho de investigação “Object-Process Linking and Embedding” (OPLE) en o caso da ensino da aritmética dos inteiros, desenvolvida por Linchevski e Williams (1999) na tradição da Educação Matemática Realista (realistic mathematics education (RME)). Nossa análise aplica a teoria da objetivação de Radford, com o propósito de surgir novas pistas sobre a forma em que a reificação tem lugar. Em particular, o método de análise mostra como a generalização factual da estratégia chamada de compensação encapsula a noção de “agregar de um lado é o mesmo que quitar do outro lado”; uma base fundamental disso que será, mais tarde, as operações com inteiros. Discutimos, de igual modo, outros aspectos da objetivação suscetíveis de chegar a ser importante na cadeia semiótica que os alunos executam na seqüência OPLE, seqüência que pode levar a um fundamento intuitivo das operações com os inteiros. Sustentamos que é necessário elaborar teorias semióticas para compreender o papel vital dos modelos e da modelação na implementação das reificações no seio da Educação Matemática Realista (RME).

● **PALAVRAS CHAVE:** Inteiros, Semióticos, Teoria de Aprendizagem.



## RÉSUMÉ

Nous rapportons ici l’analyse d’une expérience qui vise à reproduire le travail de recherche “Object-Process Linking and Embedding” (OPLE) dans le cas de l’enseignement de l’arithmétique des entiers développé par Linchevski et Williams (1999) dans la tradition de l’Éducation Mathématique Réaliste (realistic mathematics education (RME)). Notre analyse applique la théorie de l’objectivation sémiotique de Radford afin d’apporter de nouveaux éclairages sur la façon dont la réification est accomplie. La méthode d’analyse montre, en particulier, comment la généralisation factuelle de la stratégie appelée de compensation encapsule la notion que « ajouter d’un côté, c’est la même chose qu’enlever de l’autre côté » : une base fondamentale de ce que sera plus tard les opérations avec des entiers. Nous discutons également d’autres aspects de l’objectivation susceptibles de devenir importants dans la chaîne sémiotique que les élèves accomplissent dans la séquence OPLE, séquence qui peut mener à un fondement intuitif des opérations sur des entiers. Nous soutenons qu’il est nécessaire d’élaborer des théorisations sémiotiques pour comprendre le rôle vital des modèles et de la modélisation dans l’implémentation des réifications au sein de l’Éducation Mathématique Réaliste (RME).

● **MOTS CLÉS:** Entiers, sémiotique, théories de l’apprentissage.

## The Need for a Semiotic Analysis

Based on the instructional methodology of *Object-Process Linking and Embedding (OPLE)* (Linchevski & Williams, 1999; Williams & Linchevski, 1997), the *dice games instruction method* for integer addition and subtraction showed how students could intuitively construct integer operations. This methodology, underpinned by the theory of *reification* (Sfard, 1991; Sfard & Linchevski, 1994), was developed within the Realistic Mathematics Education (RME) instructional framework. Until very recently, the dice games method had not been analysed semiotically. We believe a semiotic analysis of students' activities in the dice games will illuminate students' meaning-making processes. It will also provide some further understanding of the reification of integers in the dice games in particular and more generally of the theory of reification, which does not explain "what spur[s] the students to make the transitions between stages" (Goodson-Espy, 1998, p. 234). Finally, it will contribute to the discussion of the semiotic processes involved in RME, which are currently insufficiently investigated (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). In this paper we focus on the *compensation strategy* (Linchevski & Williams, 1999), a dice game strategy on which integer addition and subtraction are grounded, and begin to address the following questions:

1. What are the students' semiotic processes of the compensation strategy in the reification of integers through the OPLE teaching of integers in the dice games method?
2. What is the semiotic role of the abacus in the OPLE teaching of integers

through the dice games and what can we generally hypothesise about the significance of models and modelling in the RME tradition?

We found Radford's semiotic theory of *objectification* (Radford, 2002, 2003) to be a particularly useful theoretical framework for analysing students' semiotic processes in the dice games, despite the very different context in which it was developed.

## The Object-Process Linking and Embedding Methodology

Sfard (1991) reported as follows:

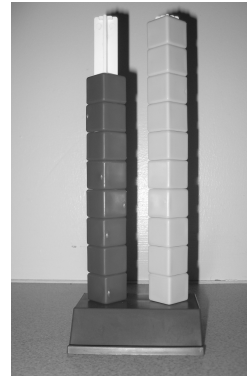
But here is a vicious circle: on the one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization – without such objects the processes must appear quite meaningless. In other words: *the lower-level reification and the higher-level interiorization are prerequisites of each other!* (p. 31)

In order to overcome this 'vicious circle', the *Object-Process Linking and Embedding (OPLE)* pedagogy (Linchevski & Williams, 1999) was developed: "children a) build strategies in the situation, b) attach these to the new numbers to be discovered, and finally c) embed them in mathematics by introducing the mathematical voice and signs" (Linchevski & Williams, 1999, p. 144). The pedagogy can be best understood through the dice games context in which it was developed (Linchevski & Williams, 1999), which

aimed at overcoming the paradox of reification described above for the case of arithmetic of the integers.

The *dice games instruction method* (Linchevski & Williams, 1999) is an intuitive instruction of integer addition and subtraction in the RME instructional framework aiming at the reification of integers. The transition from the narrower domain of natural numbers to the broader domain of integers in the method is achieved through *emergent modelling* (Gravemeijer, 1997a, 1997b, 1997c; Gravemeijer et al., 2000) and takes advantage of students' *intuition of fairness* (Liebeck, 1990) for the cancellation of negative amounts by equal positive amounts (Dirks, 1984; Linchevski & Williams, 1999; Lytle, 1994). Practically, the model – the double abacus (see figure 1) – affords the representation and manipulation of integers as objects before they are abstracted and symbolised as such by the students (Linchevski & Williams, 1999):

The integer is identifiable in the children's activity first as a process on the numbers already understood by the children, then as a 'report' or score recorded (concretised by the abacus). The operations on the integers arise as actions on their abacus representations, then recorded in mathematical signs. Finally, the operations on the mathematical signs are encountered in themselves, and justified by the abacus manipulations and games they represent. Thus the integers are encountered as objects in social activity, *before* they are symbolised mathematically, thus intuitively filling the gap formerly considered a major obstacle to reification. (Linchevski & Williams, 1999, p. 144)



**Figure 1:** *The abacus*

Therefore, in the games the situated strategies are constructed in a realistic context which allows intuitions to arise. In this process the abacus model is utilized which “affords representation of the two kinds of numbers, and allows addition and subtraction (though clearly not multiplication and division) of the integers to be based on an extension of the children's existing cardinal schemes” (Linchevski & Williams, 1999, p. 135). These strategies are linked to objects (yellow and red team points, see next section), thus allowing object-process linking. Later, the formal mathematical language and symbols enter the games. In the following section we present the games more analytically.

### The Dice Games Instruction

The method involves 4 games in each of which two teams of two children are throwing dice (e.g. a yellow and a red die in game 1) and recording team points on abacuses: the points for the yellow team are recorded by yellow cubes on the abacuses and those for the red team are red cubes on the abacuses. The students sit in two pairs, each having a member of each team and an abacus

(see figure 2). On each pair's abacus, points for both teams are being recorded and the team points on the two abacuses add up. The students in turn throw the pair of dice, recording each time the points for the two teams on their abacus. When the two abacuses combine to give one team a score of 5 points ahead of their opponents, that team wins the game. For instance in game 1, if the yellow team at a certain point is 2 ahead and they get a score on the pair of dice, say 4 yellows and one red, then they can add 3 yellows to their existing score of 2 and so get 5 ahead, and they win. But note the complication that because we have two abacuses for the two pairs, a 'combined score of 2 yellows' might involve, say 1 red ahead on the one abacus and 3 yellows ahead on the other abacus: so there are multiple 'compensations' of reds and yellows going on in various combinations. Therefore, the important thing in the games is not how many points a team has, but how many points *ahead* of the opponent: hence the nascent directivity of the numbers.

In the first game (game 1) two dice are used, a yellow and a red one, giving points to the yellow and red team in each throw. Shortly after the beginning of game 1, often with the urging of the researcher, the students intuitively understand that they can cancel the team points on the dice, thus introducing an important game strategy, the *cancellation strategy* (not examined in this paper). For example, according to this strategy, if a throw of the pair of dice shows 3 points for the yellow team and 1 for the red team, this is equivalent to just giving 2 points for the yellow team. The rationale is that the *directed difference* of the points of the yellow and red team (i.e. the amount of points that the yellows are ahead or behind the reds) will be the same anyway. As the abacus columns have only space for 10 points for each team, a team column will often be full before a team gets 5 points ahead of the opponent. In order for the game to go on, the *compensation strategy* is formulated, that is, if you can't add points to one team, subtract the same amount of points from the other, so as to maintain the correct directed difference of team points.



**Figure 2:** Students playing one of the dice games

This strategy is the second important game strategy and it is the one focused upon in this paper. By the end of the games, this strategy will lead to the intuitive construction of equivalences like:  $+(+2) \equiv -(-2)$  and  $+(-2) \equiv -(+2)$ .

Game 2 is similar to game 1, and is introduced as soon as (and not before) the children are able to cancel the pair of dice into ONE score quite fluently. In this game an extra die is now thrown whose faces are marked 'add' and 'sub' (subtract). From now on this will be called the *add/sub die*. The introduction of this die allows for subtraction to come into play, instead of just addition, as in game 1. In analogy to game 1, according to the compensation strategy, if you have to subtract points from a team but there are none on the abacus to subtract, you can add points to the other team instead.

In game 3, formal mathematical symbols for integers are introduced. The *add/sub die* is not used and the yellow and red die are replaced with an *integer die* giving one of the following results on each throw:  $-1, -2, -3, +1, +2, +3$ . Positive integers are points for the yellow team and negative integers are points taken from the yellows, thus they are points for the reds (for more details see Linchevski & Williams, 1999). Here the mathematical voice is encouraged, so that the children say "minus 3" and "plus 2" etc.

In the final game (game 4), the *add/sub die* is back into the game, allowing again for subtraction to be concerned. In these two games the cancellation strategy is no longer needed and the compensation strategy is transformed into a formal symbolic, though still verbal, form: "add minus 3" etc. Once the students become fluent in game 4, they begin recording the games for a transition from verbal to written

use of formal mathematical symbols, but we are not going to discuss this transition further in this paper.

### Some Earlier Analyses: Reification in the Dice Games

Linchevski and Williams (1999) have analysed the dice games in terms of reification. Through the instructional methodology of Object-Process Linking and Embedding, they achieved the intuitive reification of integers and the construction of processes related to integer addition and subtraction through the manipulation of objects on a model (i.e. the yellow and red team points). However, they did not provide a semiotic-analytical account of the reification processes – their main concern was to show that reification of integers was possible through their method. We will discuss here the reifications taking place in the dice games, as we understand them, so that we can better appreciate the need for a semiotic analysis of students' processes.

In relation to the reification of integers, according to Linchevski & Williams (1999), the object-process linking allows the intuitive manipulation of integers as objects from the very beginning of the dice games. As a result of this methodological innovation, some elementary processes are obvious from the beginning. These are, that if a team gets points (or points are subtracted from it), the new points add-up to (or are subtracted from) the points the team already has. These processes are intuitively obvious from the introduction of game 1 (and game 2 respectively). However, one may argue that the students still operate at the level of natural numbers, not integers.

Integer processes begin to be constructed, though integers are not yet introduced

explicitly, once the students focus on the *score* of the game, that is, which team is ahead and by how many points. The calculation of the *score* as the *directed difference* of the piles of cubes of the points of the two teams is the first *object-process link* to be constructed. The second *object-process link* to be achieved in the games is the cancellation of the team points on the dice: i.e. if in a throw the yellows get 2 points and the reds 1 point, you might as well just give 1 point to the yellows. Thus this link is possible through the establishment of the so called *cancellation strategy*. Further, the *compensation strategy* – according to which *adding to one side of the abacus is the same as subtracting from the other side* – needs to be introduced as an *object-process link*. Up to this point, all the necessary object-process links are in place. Next, at the beginning of game 3, integers are introduced into the games: the formal mathematical voice enters the games. Through the manipulation of the formal mathematical symbols of integers in the above object-process links, integers are being reified and the addition and subtraction of integers are being established.

However, in the above analysis the following significant question arises: What are the meaning-making processes (semiotic) involved in students' integer reification in the dice games? We certainly do not claim that we will exhaust this issue here, but we will begin to address it through the vital component of the compensation strategy.

### **Semiotics are Needed to Complement Reification Analyses**

The theory of reification, drawing support from a cognitivist/constructivist view of learning, is mainly interested in the internal processes of students' abstraction of

mathematical objects. It does not generally refer to the social semiotic means students used to achieve the abstraction of these objects, (e.g. in the dice games, the integers). The analysis of Linchevski & Williams (1999) did in fact go some way in providing a social analysis of the context as a resource for construction of the compensation strategy: they were excited mainly here by the accessing of the socio-cultural resource of 'fairness' in the games as a basis for an intuitive construction of compensation. Semiotic chaining was adduced to explain the significance of the transition to the 'mathematical voice', so that "two points from you is the same as two points to us" slides under a new formulation like "subtract minus two is the same as adding two... plus two". However, we will complement Linchevski & Williams' (1999) study with a more detailed semiotic analysis of the way that the abacus, gesture and deictics mediate children's generalisations (after Radford's, 2003, 2005 methodology).

We wish to clarify at this point that we do not reject the reification analyses. Instead, we agree with Cobb (1994) who takes an approach of *theoretical pragmatism*, suggesting that we should focus on "what various perspectives might have to offer relative to the problems or issues at hand" (p. 18). We propose that in this sense semiotic social theories can be complementary to constructivist ones. More precisely, we propose that Radford's theory of objectification (Radford, 2002, 2003) can be seen as complementary to the theory of reification (Sfard, 1991; Sfard & Linchevski, 1994): while Sfard (1991) provides a model for the cognitive changes taking place, Radford (2002, 2003) provides the means to analyse these changes on the social, 'intermental' plane.

Radford addresses the issue of semiotic mediation through his theory of

objectification (Radford, 2002, 2003). This theory, presented in some detail in the following section, analyses students' dependence on the available *semiotic means of objectification (SMO)* (Radford, 2002, 2003) to achieve increasingly socially-distanced levels of generality. Radford explains this reliance on SMO through reference to Frege's triad: the *reference* (the object of knowledge), the *sense* and the *sign* (Radford, 2002). The SMO refer to Frege's *sense*, that is, they mediate the transition from the *reference* to the *sign*. Moreover, Radford extended the Piagetian *schema* concept to include a *sensual dimension*, as Piaget's emphasis on the process of reflective abstraction can lead to an inadequate analysis of the role of signs and symbols (Radford, 2005).

The schema ...is ... both a sensual and an intellectual action or a complex of actions. In its intellectual dimension it is embedded in the theoretical categories of the culture. In its sensual dimension, it is executed or carried out in accordance to the technology of *semiotic activity*... (Radford, 2005, p. 7)

Given this extended schema definition, the process of abstraction of a new mathematical object needs to be investigated in relation to the semiotic activity mediating it. This investigation should expose students' meaning making processes in the objectifications taking place in the dice games, which allow the construction of integers as new mathematical objects, i.e. their reification in Sfard's sense.

In the next section we present analytically Radford's *theory of objectification* (Radford, 2002, 2003), which will then be applied in the section following it to some of our data from the instruction through the dice games.

### Radford's Semiotic Theory of Objectification

Objectification is "a process aimed at bringing something in front of someone's attention or view" (Radford, 2002, p.15). It appears in three modes of generalization: generalization through actions, through language and through mathematical symbols. These are *factual, contextual and symbolic generalization* (Radford, 2003). Objectification during these generalizations is carried out gradually through the use of *semiotic means of objectification* (Radford, 2002):

...objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (Radford, 2003, p. 41)

Factual generalization, a generalization of actions (but not of objects), is described as follows:

... A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level (e.g., "1 plus 2, 2 plus 3" ...). In addition, this scheme enables the students to tackle virtually any particular case successfully. (Radford, 2003, p. 47)

The formulation of the operational scheme of factual generalization is based on deictic semiotic activity, e.g. deictic gestures, deictic linguistic terms and rhythm. The students rely



on the signification power provided by deictics to refer to actions on non-generic physical objects. These are perceivable, non-abstract objects which can be manipulated accordingly. In the example from Radford (2003) below, the students had to find the number of toothpicks for any figure in the following pattern.

The elaboration of the operational scheme in this case can be seen in the following section of an episode provided by Radford (2003).

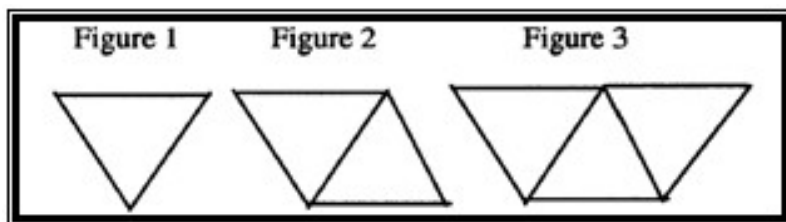
1. Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 [...]. (Radford, 2003, p. 46-47)

Josh constructed the operational scheme for the calculation of the toothpicks of any figure in the form "1 plus 2, 2 plus 3", while pointing to the figures. Moreover, he used the linguistic term *always* to show the general applicability of this calculation method for any specific figure and the term *next* which "emphasizes the ordered position of objects in the space and shapes a perception relating the number of toothpicks of the next figure to the number of toothpicks in the previous figure" (p. 48). Hence, in factual generalization:

...the students' construction of meaning has been grounded in a type of social understanding based on implicit agreements and mutual comprehension

that would be impossible in a nonface-to-face interaction. ... Naturally, some means of objectification may be powerful enough to reveal the individuals' intentions and to carry them through the course of achieving a certain goal. (Radford, 2003, p. 50)

In contextual generalization the previously constructed operational scheme is generalised through language. Its generative capacity lies in allowing the emergence of new abstract objects to replace the previously used specific concrete objects. This is the first difference between contextual and factual generalization: new abstract objects are introduced (Radford, 2003). Its second difference is that students' explanations should be comprehensible to a "generic addressee" (Radford, 2003, p. 50): reliance on face-to-face communication is excluded. Consequently, contextual generalization reaches a higher level of generality. More specifically, in Radford (2003) the operational scheme "1 plus 2, 2 plus 3" presented above becomes "You add the figure and the next figure" (p. 52). Therefore, the pairs of specific succeeding figures 1, 2 or 2, 3 become *the figure* and *the next figure*. These two linguistic terms allow for the emergence of two new abstract objects, still situated, spatial and temporal (Radford, 2003). Reliance on face-to-face communication is eliminated, and deictic means subside. However, the personal voice, reflected through the word *you*, still remains.



**Figure 3:** First three 'Figures' of the 'toothpick pattern', labelled 'Figure 1', 'Figure 2', 'Figure 3' by Radford (the picture in the box was taken from Radford, 2003, p. 45)

In symbolic generalization, the spatial and temporal limitations of the objects of contextual generalization have to be withdrawn. Symbolic mathematical objects (in Radford's case algebraic ones) should become "nonsituated and nontemporal" (Radford, 2003, p.55) and the students lose any reference point to the objects. To accomplish these changes, Radford's (2003) students excluded the personal voice (such as *you*) from their generalization and replaced the generic linguistic terms *the figure* and *the next figure* with the symbolic expressions  $n$  and  $(n+1)$  correspondingly. Hence, the expression *you add the figure and the next figure* became  $n+(n+1)$ . Still, Radford (2003) points out that for the students the symbolic expressions  $n$  and  $(n+1)$  remained indexed to the situated objects they substituted. This is evident in students' persistent use of brackets and their refusal to see the equivalence of the expressions  $n+(n+1)$  and  $(n+n)+1$ . Summarising, the mathematical symbols of symbolic generalization were indexes of the linguistic objects of contextual generalization, which in turn were indexes of the actions on concrete physical objects enclosed in the factual generalization operational scheme.

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#### **The Compensation Strategy – Factual Generalization**

In this section we analyse the objectification of the compensation strategy in terms of factual, contextual and symbolic generalization. We present excerpts of the discourse contained in the games, which we analyse in terms of their contribution to the progressive abstraction of integers through the means of objectification. We also discuss the SMO involved in students' processes. The analyses of factual, contextual and symbolic generalization are presented

separately, but first we provide some information about the students and the episodes in this paper.

The study, part of an ongoing PhD research, involves year 5 students in Greater Manchester, who had not yet been taught integer addition and subtraction. The PhD involves two experimental methods (respectively containing 5 and 6 groups of 4 students) from 2 separate classes and a control group from a third class. In each experimental method class the students were arranged by their teacher in mixed gender and ability groups, which were taught for three one-hour lessons. In this paper we focused on a microanalysis of one group of one of the methods – the dice games as originally applied by Linchevski and Williams (1999).

Radford's factual generalization is quite a clear-cut process based on action on physical objects formulated into an operational scheme through deictic activity. However, in our investigation of the compensation strategy, we find a multi-step process of semiotic contraction happening inside it. The three following episodes co-constitute in our view the factual generalization. In these episodes, occurring during game 1 (in lesson 1), the students were faced with a situation where they had to add cubes/points to one of the two teams, but there was no space on the abacus. As a result, a breakthrough was needed for the scoring to continue.

Episode 1 (Minutes 14:30-14:50, lesson 1): Umar had to add 1 yellow cube on the abacus but, as there was no space in the relevant column, he got stuck. Fay proposed taking away 1 red cube instead. "... " indicates a pause of 3 sec or more, and "." or "," indicate a pause of less than 3 sec" (Radford, 2003, p. 46).

Fay: You take 1 off the reds [*pointing to the red column on her abacus*]. [...] Because then you still got the same, because you're going back down [*showing with both her hands going down at the same level*] 'cause instead of the yellows getting one [*raising the right hand at a higher level than her left hand*] the red have one taken off [*raising her left hand and immediately moving it down, to show that this time the reds decrease*].

Fay's proposal for the subtraction of a red cube instead of the addition of a yellow one is the first articulation of the compensation strategy in the games for this group of students. We especially noticed the analytical explanation of the proposed action, which allows the process of compensation to be introduced for the first time. Deictic activity was associated both with the proposed action of taking away a red cube and with the justification following it. Fay used pointing to the red cubes on the abacus, as well as a gesture with both her hands indicating the increase/decrease of the pile of cubes in each team's column. Moreover, the names "the yellows" and "the red" have a deictic role. We also notice the phrase "you still got the same", stressing that something (obviously important) remains unaltered: either we add a yellow

point/cube or subtract a red point/cube. This significant unaltered game characteristic, which we call the *directed difference* of the points of the yellow and red team, still cannot be articulated as it has not yet acquired a name.

Episode 2 (Minutes 20:15-20:43, lesson 1): The yellows' column was full and the reds' only had space for 1 cube. Compensation was needed and as Zenon could not understand, Jackie explained as follows.

Jackie: It's still the same, like ... [*a very characteristic gesture (see figure 4): she brings her hands to the same level and then she begins to move them up and down in opposite directions, indicating the different resulting heights of the cubes of the two columns of the abacus*] because it's still 2, the yellows are still 2 ahead [*she does the same gesture while she talks*] and the reds are still 2 below, so it's still the same... [*again the gesture*] ... em like... [*closing her eyes, frowning hard*] ... I don't know what it's called but it's still the same... score [*the gesture 'same' again before and while articulating the word "score" – indicating 'same' score on her abacus*].



Figure 4: Jackie's gesture (this sequence of action performed fast and repeated several times)

In episode 2, we noticed the repeated use of the phrase “it’s still the same”, the word “still” followed by the difference in team points (i.e. “still 2”, “still 2 ahead”), as well as the accompanying characteristic gesture. The gesture, too, emphasized the importance of the unaltered directed difference of the cubes of the two teams. We also noticed Jackie’s difficulty in finding a proper word for this important unaltered game characteristic: “em like... [closing her eyes, frowning hard] ... I don’t know what it’s called but it’s still the same... score” (extract from episode 2 above). We believe the articulation of the word *score*, meaning what we call the directed difference of team points, as well as Jackie’s gesture were very important for the factual generalization process, because they achieved the *semiotic contraction* (Radford, 2002) of the process originally established in episode 1. From this point onward, the students do not need to provide an analytical semiotic justification of the proposed action, as Fay needed to in episode 1. Just saying that the score will be the same is enough. A similar effect was accomplished by the word *difference* in a different group (Koukkoufis & Williams, 2005).

Episode 3 (Minutes 21:27-21:57, lesson 1):

There’s only space for 2 yellow cubes, but Fay has to add 3 yellows and 1 red.

Fay: Add 2 on [*she adds 2 yellow cubes*] and then take 1 of theirs off [*she takes off a red cube*] and then for the reds [*pointing to the red dice*] you add 1, so you add the red back on [*she adds 1 red cube*].

Researcher: [...] Does everybody agree? (Jackie and Umar say “Yeah”).

Finally, in the above episode further semiotic contraction took place. In fact, no

justification of the proposed action was provided, as it seemed to be unnecessary – indeed Jackie and Umar agreed with Fay without further explanation. We argue that the further semiotic contraction happening in episode 3 completed the factual generalization of the compensation strategy.

To sum up, we see in the three episodes provided up to this point a continuum as follows: in episode 1 Fay presented a proper action and an analytical process to justify it; in episode 2 again a proper action was presented but the process justifying it was contracted; finally in episode 3 the presentation of the proposed action was sufficient, therefore further semiotic contraction took place and the process for resulting in this proposed action disappeared.

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### **The Compensation Strategy – Contextual Generalization**

Contextual generalization, in which abstraction of new objects through *language* takes place, has not yet been completed in this case. If we had had a contextual generalization of the compensation strategy, we would have a generalization like this: if you can’t add a number of yellow/red points, you can subtract the same number of red/yellow points instead. Similarly for subtraction, the generalization would be similar to this: if you can’t subtract a number of yellow points/red points, you can add the same number of red/yellow points instead. However, our students did not spontaneously produce such a generalization, neither does the instructional method demand it; therefore we did not insist that the students produce it. We believe that the lack of articulation of the compensation strategy through

generic linguistic terms, and thus the incompleteness of the production of a contextual generalization, has to do with the compensation strategy being too intuitively obvious. On the contrary, in the case of the *cancellation strategy* (Linchevski & Williams, 1999) which was not so obvious, the same students produced a contextual generalization as follows (Fay, minutes 38:17-38:40, lesson 1, 5 reds and 2 yellows): “you find the biggest number, then you take off the smaller number”. In the case of the contextual generalization of the cancellation strategy, we notice that new abstract objects (“the biggest number”, “the smaller number”) enter the discourse, as in Radford (2003). However, we will not discuss the contextual objectification of the cancellation strategy here.



### **The Compensation Strategy – Symbolic Generalization**

Despite the incompleteness of the contextual generalization, we found that symbolic generalization was not obstructed! In this section we discuss the symbolic generalization of the compensation strategy, which presents some differences from that of the case presented by Radford (2003).

To begin with, in Radford (2003) symbolic generalization remained indexical throughout the instruction. In our case, the students began using symbolic generalization non-indexically. For convenience, we present indexical and non-indexical symbolic generalization separately.

#### *Indexical Symbolic Generalization*

The elaboration of a symbolic generalization for the compensation strategy demands the replacement of pre-symbolic signs with symbolic ones.

Therefore, the reference to yellow and red team points has to be substituted by reference to positive and negative integers. According to the dice games method, this is achieved in the beginning of game 3, when *the red and the yellow die* are replaced by the *integer die*. Analytically, the numbers +1, +2 and +3 (on the integer die) are points for the yellow team. Further, -1, -2 and -3 (on the integer die) are points taken away from the yellow team, thus they are points for the red team. Of course, similarly one can say that +1, +2 and +3 are point taken away from the red team. Conclusively, when it is “+” it is yellow points, while when it is “-” it is red points. In the following episode we witness the transition from the pre-symbolic signs of “yellow team points” and “red team points” to the symbolic signs of “+” and “-” (positive and negative integers).

Episode 4: Minutes: 20:45-21:55, lesson 3.

Researcher: +1. Who is getting points?  
 Jackie: The yellows  
 Researcher: [...] Who is losing points?  
 Jackie, Umar: The reds  
 Fay: [...] reds are becoming called minuses and then the yellows are becoming called plus.

As a result of the above introduction of the formal mathematical symbols for integers, positive integers are used to indicate yellow team points and negative integers are used to indicate red team points. Here lies the first difference from Radford’s symbolic generalization, which is soon to become evident.

In Radford (2003), the symbolic signs/expressions used in symbolic generalization were indexes of the contextual abstract objects of contextual generalization. Hence, the expressions //

and  $n + 1$  indicated the generic linguistic terms *the figure* and *the next figure*. Instead, in the dice games the formal mathematical symbols of integers were indexes not of the generic linguistic terms of contextual generalization (which was never completed), but of the concrete objects of factual generalization. For example,  $+2$  is an index of “2 more for yellows” as well as of “2 yellow points”, as in episode 5.

#### Episode 5 (Minutes: 33:15-33:53, lesson 3)

- Researcher: [...] you get  $-2$ . What would you do? (Fay takes 2 yellow cubes off) [...] What if you had  $+3$ ?
- Umar: You take away 3 of the reds.
- Zenon: ... or you could add 3 to the yellow.
- Fay, Jackie: ... add 3 to the yellow.
- Researcher: Oh, 3 off the reds or 3 to the yellows. (All the students agree)

Indeed, the students read  $+2$  on the die, the researcher articulates it as “plus 2”, but then the students’ discussion is in terms of reds and yellows. If symbolic signs were being used non-indexically at that point, Umar would have said “minus 3” instead of saying “3 of the reds” (as in the phrase “take away 3 of the reds”). Also the others would have said “plus 3” instead of “3 to the yellow” (as in the phrase “add 3 to the yellow”). It becomes clear that in our case, we witnessed a *direct* transition from *factual* to *indexical symbolic generalization*, without the completion of contextual generalization being necessary. This transition was afforded due to the RME context and the abacus model.

In indexical symbolic generalization, though the operational scheme of factual

generalization is reconstructed through the use of symbolic signs instead of concrete physical objects, it is not a simple repetition of factual generalization in symbolic terms that takes place. No semiotic contraction needs to take place for the establishment of the compensation strategy in symbolic terms. The students know right away that instead of adding  $+2$  (2 yellow points) they can subtract 2 red points.

#### *Non-indexical Symbolic Generalization*

Up to now the formal symbolic signs for integers are being used indexically, but the intended instructional outcome is that students will eventually be using these symbols non-indexically. We do not imply that the symbols should drop their connection to the context though. Indeed it is essential that students can go back to the contextual meanings of these symbols in the dice games, so as to draw intuitive support regarding integers. We just emphasize that the students should become flexible in using the formal symbols of integers either indexically or non-indexically. A non-indexical use of integer symbols would mean explicit reference solely to *pluses* and *minuses* (i.e.  $+2$ ,  $-3$  etc). Therefore, the compensation strategy should be constructed only based on the formal symbols of integers, excluding the pre-symbolic signs of yellow and red team points.

In order to target *non-indexical symbolic generalization*, we encouraged students to articulate the symbols on the dice as “+” (*plus*) and “-” (*minus*), in an attempt to facilitate the connection of the verbalization *plus/minus* to the symbolic signs  $+/-$ . Though in the beginning most students needed to be reminded to use the “proper” names of the signs, by the time the students had played game 4 for a while they were

able to refer to integers in a formal manner, as can be seen in the following examples of student verbalizations. We believe that the introduction of the *add/sub die* in game 4 obliged the students to refer correctly to the integers with their formal names, so as to be able to perform the actions of addition and subtraction on these symbols. For example (brackets added), Fay said: *add [minus 3], subtract [2 of the minuses]*; Zenon said: *add [2 to the pluses]*; Jackie said: *add [minus 2]*. Umar was still struggling with the verbalization and sometimes said *[minus 1] add* or *add [subtract 2]* etc.

Finally, we checked if students had spontaneously produced a more general verbalization in a form like “if you can’t add pluses/minuses, you can subtract minuses/pluses” or the other way around. In this group, such a generalization did not take place. We believe, however, that this will not necessarily be the case for other groups of students, and indeed that it may be desirable to encourage this in the teaching.

### The Semiotic Role of the Abacus Model

As may be clear by now, the abacus model and the RME context of the dice games are very significant for the reification of integers and the instruction of integer addition and subtraction through the dice games method. Up to now we have referred to the semiotic processes, but we have not referred to the abacus model: though Radford’s theory of objectification has been crucial in the analyses so far, we contend it needs to be complemented by an analysis of the role of the abacus in affording these semiotics. We claim that analysing the contribution of the model in students’ semiosis will afford some primary discussion of phenomena such as (i) the embodiment of semiotic activity, (ii) the

incompleteness of the contextual generalization and (iii) the direct transition from factual to symbolic generalization.

The abacus model in the games seems in many ways to be the centre of the activity: the abacus is in the centre of a ‘circle of attention’, as we are all sitting around the abacuses (see figure 2 again); it affords the representation of the yellow and red team points through their red and yellow cubes; it is the constant point of reference about which team is ahead. It was only natural that the abacus, being in the centre of the spatial arrangement and credited with allowing the students to keep the score, became the focus of semiotic activity. What is even more important: the abacus mediated in some cases the semiotic activity.

This can be seen in several features of the games. To begin with, the team points referred to the above episodes as “points for the yellow/red” (or as “yellow/red points”) were concretized or ‘objectified’ from the start: they were yellow and red cubes. That is, *the points were embodied into the cubes*. This allows, as Linchevski and Williams (1999) point out, for integers to be introduced in the discourse as objects from the very beginning: the students speak about the general categories of yellow and red points from the beginning. Additionally, the *directed difference was embodied on the abacus*, as the difference of yellow and red points can be seen with a glance at the abacus, and the sign is evidently that of the larger pile of cubes: i.e. in figure 1 the yellows on that abacus are 2 points ahead. This convenient reference to the directed difference in the two piles of cubes afforded the association of semiotic activity to it, which made the establishment of the compensation strategy possible. Such semiotic activity is Fay’s gesture in episode 1 in which the

movements of her hands were matched with a verbal manipulation of the difference of the two piles of cubes (i.e. “you’re going back down”) to show that the directed difference remained the same. Also Jackie’s quick movement of her hands up and down in episode 2 again indicates the difference in the two piles of cubes, in other words it points to the directed difference as it is embodied on the abacus. We suggest that this embodiment is crucial because it mediates the emergence of deictic semiotic activity such as that of Fay and Jackie in episodes 1 and 2 and hence allows the objectification to take place. We may even consider the toothpick figures in Radford (2003) to afford the same role.

As we noted above, the embodiment of the yellow and red team points through the cubes allowed the introduction of points for the yellow and points for the red as general abstract categories. We mentioned earlier that the students did not complete the contextual generalization of the compensation strategy to produce a generalization like “if you can’t add a number of yellow/red points, you can subtract the same number of red/yellow points instead”. However, the embodiment of the yellow and red team points of the abacus had already introduced generic situated objects into the discourse, even though this was not achieved through language. Consequently, the students could obviously see that the operational scheme of the factual generalization can be applied for any number of points for a team. This is an additional reason to the one presented earlier for the incompleteness of contextual generalization. Hence, this embodiment of the team points in a sense shapes the semiotic activity in the compensation strategy, providing one more reason why the completeness of contextual generalization was unnecessary in this case.

Further, the semiotic role of the abacus was crucial in the direct transition from factual to symbolic generalization. As we have seen in episode 4, the yellow points became “plus” and the red points are now “minuses”. We say that this direct transition was afforded through the construction of a *chain of signification* (Gravemeijer et al., 2000; Walkerdine, 1988), in the form of a transition from the embodiment of yellow and red points through the abacus cubes to the embodiment of positive and negative integers. As a consequence of this transition, the formal symbols could be embedded into the operational scheme for the compensation strategy established through the factual generalization. Quite naturally then, the embedding of the formal symbols in the operational scheme performed on the abacus produced the symbolic generalization directly from factual generalization.

## ● Conclusion

Beginning with a presentation of the OPLE methodology and the dice games instruction, we argued the need for a finer grained, semiotic analysis of objectifications to explain how reification is accomplished.

We have applied Radford’s theory of objectification to fill this gap in understanding the case of the compensation strategy, a vital link in the chain of significations necessary to OPLE’s success: thus we were able to identify relevant objectifications applying Radford’s semiotic categories of generalisation. This work began to reveal the significance of the abacus itself, which affords, and indeed shapes the semiosis in essential ways. We have also shown how the effectiveness of the pedagogy based on OPLE can be explained as semiotic chaining using multiple semiotic objectifications and begun to discuss the



significance of models and modelling in the dice games, and hence in the OPLE methodology. Finally, we suggest that our discussion over the semiotics of the abacus model might be the route to understanding the significance of models and modelling in the RME tradition more generally. We suggest that the role of the abacus as a model in this case might be typical of other models in RME. Indeed, Williams & Wake (in press) provide an analysis of the role of the number line in a similar vein.

In applying Radford's theory in a very

different context we are bound to point out certain differences in the two cases: for instance, the differing roles of contextual generalisation in the two cases. Though the adaptation of the theory was necessary at some points, we have shown that this theory can be a powerful tool of analysis. The question arises as to whether the instruction method adopted here gains or perhaps loses something by eliding contextual generalisation: thus we suggest that Radford's categories might in fact be regarded as raising design-related issues as well as providing tools of analysis.

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