Are registers of representations and problem solving processes on functions compartmentalized in students' thinking?

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RESUMEN

El objetivo de este artículo es doble. En primer lugar, se hace un resumen superficial de investigaciones sobre la compartimentación de diferentes registros de representación, así como de las aproximaciones de resolución de problemas, relacionadas con el concepto de función. En segundo lugar, se aportan elementos que clarifican las posibles maneras que permiten superar el fenómeno de la compartimentación. Investigaciones precedentes muestran que la mayoría de los alumnos de secundaria e, incluso de universidad, tienen dificultades para cambiar, de forma flexible, los sistemas de representación de funciones, de seleccionar y de utilizar aproximaciones apropiadas de resolución de problemas. Los resultados de dos estudios experimentales previos, llevados a cabo por miembros de nuestro equipo de investigación, centrados sobre la utilización de aproximaciones no tradicionales de enseñanza y sobre el empleo de software matemático, proveen pistas preliminares, en cuanto a la manera de cómo puede superarse con éxito el fenómeno de la compartimentación.

- **PALABRAS CLAVE:** Aproximación algebraica, compartimentalización, función, aproximación geométrica, resolución de problemas, registros de representación, transformación de representaciones.

ABSTRACT

The purpose of the present study is twofold: first, to review and summarize previous research on the compartmentalization of different registers of representations and problem solving approaches related to the concept of function; second, to provide insights into possible ways to overcome the phenomenon of compartmentalization. To this extent, previous research shows that the majority of high school and university students experience difficulties in flexibly changing systems of representations of function and in selecting and employing appropriate approaches to problem solving. Two previous experimental efforts, by the authors, focusing on the use of non-traditional teaching approaches and on the use of mathematical software respectively, provided some initial strategies for successfully overcoming the phenomenon of compartmentalization.

- **KEY WORDS:** Algebraic approach, compartmentalization, function, geometric approach, problem solving, registers of representation, transformation of representations.

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RESUMO

O objetivo deste de artigo é duplo. Primeiro, é feito um resumo superficial de investigações sobre a compartimentação de diferentes registros de representação, e aproximações de resolução de problemas, apostas em relação ao de conceito de função. Em segundo lugar, traz elementos que clarificam as possíveis maneiras que permitem superar o fenômeno da compartimentarão. Investigações precedentes mostram que a maioria dos alunos do ensino médio e, mesmo de universidade, tem dificuldades para alterar, de maneira flexível, os sistemas de representação de funções, de escolher e utilizar aproximações adequadas à resolução de problemas. Os resultados de dois estudos experimentais prévios, levados a efeito por membros do nosso grupo de pesquisa, centrados no utilização de aproximações não tradicionais de ensino e sobre ou emprego de «software» matemático, fornecem pistas preliminares, quanto à maneira como pode ser superar com sucesso o fenômeno da compartimentação.

- **PALAVRAS CHAVE:** Aproximação algébrica, compartimentação, função, geométrica aproximação, solução de problema, registros de representação, transformação de representações.

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RÉSUMÉ

Le but de cet article est double. En premier lieu, il s'agit de faire un survol et une synthèse des recherches sur la compartimentation de différents registres de représentation et des approches de résolution de problèmes reliées au concept de fonction. En deuxième lieu, il s'agit d'apporter un éclairage sur les manières possibles de surmonter le phénomène de compartimentation. Des recherches antérieures montrent que la majorité des élèves de l'école secondaire et de l'université ont de la difficulté à changer de façon flexible les systèmes de représentation des fonctions ainsi qu'à sélectionner et à utiliser des approches appropriées de résolution de problèmes. Deux efforts expérimentaux préalables, menés par les auteurs, centrés sur l'utilisation des approches non-traditionnelles d'enseignement et sur l'emploi de logiciels mathématiques, fournissent des indications préliminaires quant à la manière de surmonter avec succès le phénomène de compartimentation.

- **MOTS CLÉS:** Approche algébrique, compartimentation, fonction, approche géométrique, résolution de problèmes, registres de représentation, transformation de représentations.

1.INTRODUCTION

During the last decades, a great deal of attention has been given to the concept of representation and its role in the learning of mathematics. Nowadays, the centrality of multiple representations in teaching, learning and doing mathematics seems to have become widely acknowledged (D' Amore, 1998). Representational systems are fundamental for conceptual learning and determine, to a significant extent, what is learnt (Cheng, 2000). A basic reason for this emphasis is that representations are considered to be "integrated" with mathematics (Kaput, 1987). Mathematical concepts are accessible only through their semiotic representations (Duval, 2002). In certain cases, representations, such as graphs, are so closely connected with a mathematical concept, that it is difficult for the concept to be understood and acquired without the use of the corresponding representation. Any given representation, however, cannot describe thoroughly a mathematical concept, since it provides information regarding merely a part of its aspects (Gagatsis & Shiakalli, 2004). Given that each representation of a concept offers information about particular aspects of it without being able to describe it completely, the ability to use various semiotic representations for the same mathematical object (Duval, 2002) is an important component of understanding. Different representations referring to the same concept complement each other and all these together contribute to a global understanding of it (Gagatsis & Shiakalli, 2004). The use of different modes of representation and connections between them represents an initial point in mathematics education at which pupils use one symbolic system to expand and understand another (Leinhardt, Zaslavsky,

& Stain, 1990). Thus, the ability to identify and represent the same concept through different representations is considered as a prerequisite for the understanding of the particular concept (Duval, 2002; Even, 1998). Besides recognizing the same concept in multiple systems of representation, the ability to manipulate the concept with flexibility within these representations as well as the ability to "translate" the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998).

Duval (2002) assigns the term "registers" of representation to the diverse spaces of representation in mathematics and identifies four different types of registers: natural language, geometric figures, notation systems and graphic representations. Mathematical activity can be analyzed based on two types of transformations of semiotic representations, i.e. treatments and conversions. Treatments are transformations of representations, which take place within the same register that they have been formed in. Conversions are transformations of representations that consist in changing the register in which the totality or a part of the meaning of the initial representation is conserved, without changing the objects being denoted. The conversion of representations is considered as a fundamental process leading to mathematical understanding and successful problem solving (Duval, 2002). A person who can easily transfer her knowledge from one structural system of the mind to another is more likely to be successful in problem solving by using a plurality of solution strategies and regulation processes of the system for handling cognitive difficulties.

2. THE ROLE OF REPRESENTATIONS IN MATHEMATICS LEARNING: EMPIRICAL BACKGROUND

Students experience a wide range of representations from their early childhood years onward. A main reason for this is that most mathematics textbooks today make use of a variety of representations more extensively than ever before in order to promote understanding. However, a reasonable question can arise regarding the actual role of the use of representations in mathematics learning. A considerable number of recent research studies in the area of mathematics education in Cyprus and Greece investigated this question from different perspectives. In an attempt to explore more systematically the nature and the contribution of different modes of representation (i.e., pictures, number line, verbal and symbolic representations) on mathematics learning, Gagatsis and Elia (2005a) carried out a review of a number of these studies, which examined the effects of various representations on the understanding of mathematical concepts and mathematical problem solving in primary and secondary education. Many of these studies identified the difficulties that arise in the conversion from one mode of representation of a mathematical concept to another. They also revealed students' inconsistencies when dealing with relative tasks that differ in a certain feature, i.e. mode of representation. This incoherent behaviour was addressed as one of the basic features of the phenomenon of compartmentalization, which may affect mathematics learning in a negative way.

The research of Gagatsis, Shiakalli and Panaoura (2003) examined the role of the number line in second grade Cypriot students' performance in executing simple addition and subtraction operations with natural numbers. By employing implicative statistical analysis (Gras, 1996), they detected a complete compartmentalization between the students' ability to carry out addition and subtraction tasks in the symbolic form of representation and their ability to perform the same tasks by using the number line. A replication of the study by Gagatsis, Kyriakides and Panaoura (2004) with students of the same age in Cyprus, Greece and Italy, and this time using a different statistical method, namely structural equation modelling, resulted in congruent findings. This uncovers the strength of the phenomenon of compartmentalization despite differences in curricula, teaching methods, mathematics textbooks and even culture.

Michaelidou, Gagatsis and Pitta-Pantazi (2004) have examined 12-year-old students' understanding of the concept of decimal numbers based on the threefold model of the understanding of an idea, proposed by Lesh et al. (1987). To carry out the study, three tests on decimal numbers were developed. These tests aimed at investigating students' abilities to recognize and represent decimal numbers with a variety of different representations and their ability to transfer decimal numbers from the symbolic form to the number line and vice versa. The application of the implicative statistical method demonstrated a compartmentalization of students' abilities in the different tasks and this signifies that there was a lack of coordination between recognition, manipulation within a representation and conversion among different representations of decimal numbers. This finding means that some students who can recognize decimal numbers in different representations cannot use the representations to represent the decimal numbers by

themselves and, what is more important, fail to transfer from one representation of decimal numbers to another. In other words, students have not developed a unified cognitive structure concerning the concept of decimals since their ideas seemed to be partial and isolated. Given the three aspects of the understanding of mathematical concepts related to representations, namely, recognition, flexible use and conversion, it can be suggested that in this study students did not understand the concept of decimal numbers.

Finally, Marcou and Gagatsis (2003) examined 12-year-old students' understanding of the concept of fractions and more specifically the equivalence and the addition of fractions. The researchers designed three types of tests on fractions, which involved conversions among the symbolic expressions, verbal expressions and the diagrammatic representations of fractions (area of rectangles). Students' responses to the tasks were compartmentalized with respect to the starting representation of the conversions, as indicated by the implicative analysis of the data. In line with the afore mentioned studies' results, this finding means that students had a fragmentary understanding of fractions.

In the present paper, four recent studies are combined and discussed to explore secondary school and university students' abilities to use multiple modes of representation for one of the most important unifying ideas in mathematics (Romberg, Carpenter, & Fennema, 1993; Mousoulides & Gagatsis, 2004), namely functions, and to flexibly move from one representation of the concept to another. The main concern of this paper is twofold; first to identify and further clarify the appearance of the phenomenon of

compartmentalization in students' thinking about the particular concept and second to examine possible ways for succeeding at de-compartmentalization in registers of representations and problem solving processes in functions.

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3. REPRESENTATIONS AND THE CONCEPT OF FUNCTION

The concept of function is central to mathematics and its applications. It emerges from the general inclination of humans to connect two quantities, which is as ancient as mathematics itself. The didactical metaphor of this concept seems difficult, since it involves three different aspects: the epistemological dimension as expressed in the historical texts; the mathematics teachers' views and beliefs about function; and the didactical dimension which concerns students' knowledge and the restrictions imposed by the educational system (Evangelidou, Spyrou, Elia, & Gagatsis, 2004). On this basis, it seems natural for students of secondary or even tertiary education, in any country, to have difficulties in conceptualizing the notion of function. The complexity of the didactical metaphor and the understanding of the concept of function have been a main concern of mathematics educators and a major focus of attention for the mathematics education research community (Dubinsky & Harel, 1992; Sierpinska, 1992). An additional factor that influences the learning of functions is the diversity of representations related to this concept (Hitt, 1998). An important educational objective in mathematics is for pupils to identify and use efficiently various forms of representation for the same mathematical concept and to move flexibly from one system of representation of the concept to another. The influence of different representations on the understanding and interpretation of functions has been examined

by a substantial number of research studies (Hitt, 1998; Markovits, Eylon, & Bruckheimer, 1986).

Several researchers (Evangelidou et al., 2004; Gagatsis, Elia & Mougi, 2002; Gagatsis & Shiakalli 2004; Mousoulides & Gagatsis, 2004; Sfard 1992; Sierpinska 1992) indicated the significant role of different representations of function and the conversion from one representation to another on the understanding of the concept itself. Thus, the standard representational forms of the concept of function are not enough for students to be able to construct the whole meaning and grasp the whole range of its applications. Mathematics instructors, at the secondary level, have traditionally focused their instruction on the use of algebraic representations of functions. Eisenberg and Dreyfus (1991) pointed out that the way knowledge is constructed in schools mostly favours the analytic elaboration of the notion to the detriment of approaching function from the graphical point of view. Kaldrimidou and Iconomou (1998) showed that teachers and students pay much more attention to algebraic symbols and problems than to pictures and graphs. A reason for this is that, in many cases, the iconic (visual) representations cause cognitive difficulties because the perceptual analysis and synthesis of mathematical information presented implicitly in a diagram often make greater demands on a student than any other aspect of a problem (Aspinwall, Shaw, & Presmeg, 1997).

In addition, most of the aforementioned studies have shown that students tend to have difficulties in transferring information gained in one context to another (Gagatsis & Shiakalli, 2004). Sfard (1992) showed that students were unable to bridge the algebraic and graphical representations of functions, while Markovits et al. (1986) observed that

the translation from graphical to algebraic form was more difficult than the reverse. Sierpinska (1992) maintains that students have difficulties in making the connection between different representations of functions, in interpreting graphs and manipulating symbols related to functions. A possible reason for this kind of behaviour is that most instructional practices limit the representation of functions to the translation of the algebraic form of a function to its graphic form.

Lack of competence in coordinating multiple representations of the same concept can be seen as an indication of the existence of compartmentalization, which may result in inconsistencies and delays in mathematics learning at school. This particular phenomenon reveals a cognitive difficulty that arises from the need to accomplish flexible and competent translation back and forth between different modes of mathematical representations (Duval, 2002). Making use of a more general meaning of compartmentalization which does not refer necessarily to representations, Vinner and Dreyfus (1989) suggested that compartmentalization arises when an individual has two divergent, potentially contradictory schemes in her cognitive structure and pointed out that inconsistent behaviour is an indication of this phenomenon.

The first objective of this study is to identify the phenomenon of compartmentalization in secondary school students and university students' strategies for dealing with various tasks using functions on the basis of the findings of four recent research studies (Elia, Gagatsis & Gras, 2005; Gagatsis & Elia, 2005b; Mousoulides & Gagatsis, 2006; Mousoulides & Gagatsis, 2004). Although these studies explored the students' ability to handle different modes of the representation of function and move flexibly

from one representation to another, there is a fundamental difference between the mathematical activities they proposed. The study of Elia et al., (2005) investigated students' understanding of function based on their performance in mathematical activities that integrated both types of the transformation of representations proposed by Duval (2002), i.e. treatment and conversion. The study of Mousoulides and Gagatsis (2004) investigated students' performance in mathematical activities that principally involved the second type of transformations, that is, the conversion between systems of representation of the same function, and concentrated on students' approaches to the use of representations of functions and the connection with students' problem solving processes. The studies of Gagatsis et al., (2004) and Mousoulides and Gagatsis (2006) introduced two approaches that might succeed at de-compartmentalization, namely a differentiated instruction and the use of a computerized environment for solving problems in functions. Thus, what is new in this review is that students' understanding of function is explored from two distinct perspectives (which will be further clarified in the next section), but nevertheless based on the same rationale, that is, Duval's semiotic theory of representations. The second objective of the review is to discuss strategies for overcoming compartmentalization in functions.

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4. CAN WE "TRACE" THE PHENOMENON OF COMPARTMENTALIZATION BY USING THE IMPLICATIVE STATISTICAL METHOD OF ANALYSIS?

Previous empirical studies have not clarified compartmentalization in a comprehensive or systematic way. Thus,

we theorize that the implicative relations between students' responses in the administered tasks, uncovered by Gras's implicative statistical method (Gras, 1996), as well as their connections (Lerman, 1981) can be beneficial for identifying the appearance of compartmentalization in students' behaviour. To analyze the collected data of both studies, a computer software called C.H.I.C. (Classification Hiérarchique Implicative et Cohésitive) (Bodin, Coutourier, & Gras, 2000) was used.

We assume that the phenomenon of compartmentalization in the understanding of function as indicated by students' performance in tasks integrating treatment and conversion (Gagatsis & Elia, 2005b) appears when at least one of the following conditions emerges: first, when students deal inconsistently or incoherently with tasks involving the different types of representation (i.e., graphic, symbolic, verbal) of functions or conversions from one mode of representation to another; and/or second, when success in using one mode of representation or one type of conversion of function does not entail success in using another mode of representation or in another type of conversion of the same concept. As regards students' ways of approaching tasks requiring only conversions among representations of the same function (Mousoulides & Gagatsis, 2004), our conjecture is that compartmentalization appears when students deal with all of the tasks using the same approach, even though a different approach is more suitable for some of them.

4.1. Secondary school students' abilities in the transformation of representations of function (Study 1)
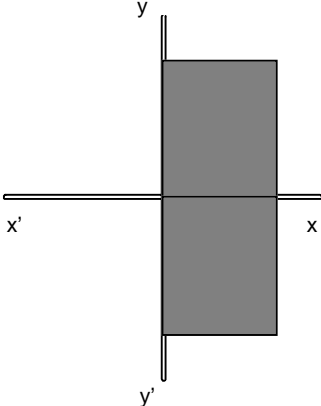
Recent studies (Gagatsis & Elia, 2005b; Elia et al., 2005) investigated secondary school students' ability to transfer

mathematical relations from one representation to another. In particular, the sample of the study consisted of 183 ninth grade students (14 years of age). Two tests, namely A and B, were developed and administered to the participants. The tasks of both tests involved conversions of the same algebraic relations, but with different starting modes of representation. Test A consisted of six tasks in which students were given the graphic representation of an algebraic relation and were asked to translate it to its verbal and symbolic forms respectively. Test B consisted of six tasks (involving the same algebraic relations as test A) in which

students were asked to translate each relation from its verbal representation to its graphical and symbolic mode. For each type of translation , the following types of algebraic relations were examined: $y < 0$, $xy > 0$, $y > x$, $y = -x$, $y = 3/2$, $y = x - 2$ based on a relevant study by Duval (1993).

The former three tasks corresponded to regions of points, while the latter three tasks corresponded to functions. Each test included an example of an algebraic relation in graphic, verbal and symbolic forms to facilitate students' understanding of what they were asked to do, as follows:

Table 1: An example of the tasks included in the test

Graphic representation	Verbal representation	Symbolic representation
	It represents the region of the points having positive abscissa.	$x>0$

It is apparent that the tasks involved conversions, which were employed either as complex coding activities or as point-to-point translations and were designed to correspond to school mathematics. However, a general use of the processes of treatment and conversion was required for the solution of these tasks. For instance, the conversion of the function $y = x - 2$ from the algebraic expression to the graphical one could be accomplished by carrying out various kinds of treatment, such as calculations in the same notation system. It is evident that in this kind of task the process of treatment cannot be easily distinguished from the process of conversion. According to this perspective, these tasks differ from the tasks proposed by Duval (1993).

The results of the study revealed that students achieved better outcomes in the conversions starting with verbal representations relative to the conversions of the

corresponding relations starting with graphic representations. In addition, all of the conversions from the graphic form of representation to the symbolic form of representation appeared to be more difficult than the conversions of the corresponding relations from the graphic form of representation to the verbal form of representation. Students perceived the latter type of conversion more easily at a level of meta-mathematical expression rather than at a level of mathematical

expression. In fact, students were asked to describe verbally (in a text) a property perceived by the graph. On the contrary, the conversions from the graphical form to the symbolic form entailed mastering algebraic concepts concerning equality or order relations as well as using the algebraic symbolism efficiently.

Figure 1 presents the similarity diagram of the tasks of Test A and Test B based on the responses of the students.
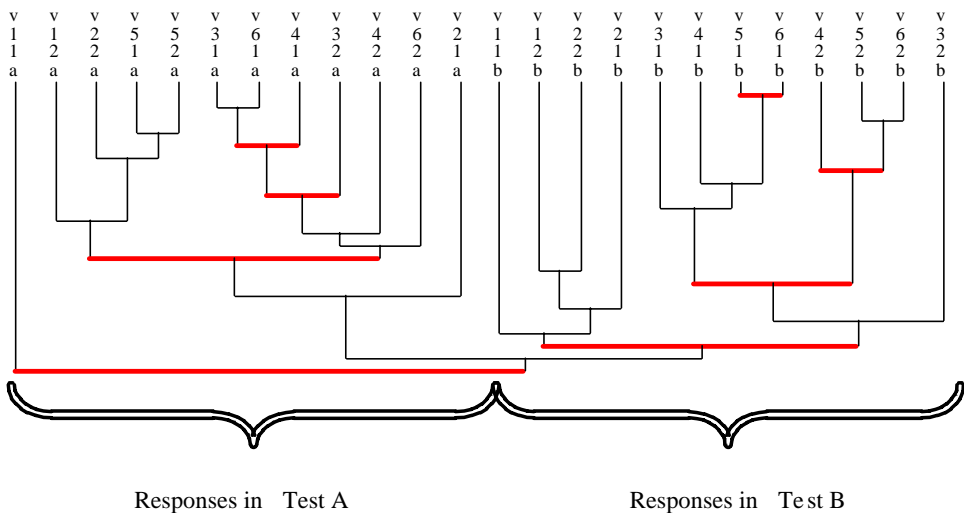

Responses in   Test A   Responses in   Te st B

Figure 1: Similarity diagram of the tasks of Test A and Test B according to Grade 9 students' responses

Note: The symbolism used for the variables of this diagram (and the diagram that follows) is explained below.

1. "a" stands for Test A, and "b" stands for Test B

2. The first number after "v" stands for the number of the task in the test
   i.e., 1: $y < 0$, 2: $xy > 0$, 3: $y > x$, 4: $y = -x$, 5: $y = 3/2$, 6: $y = x - 2$

3. The second number stands for the type of conversion in each test, i.e., for Test A, 1: graphic to verbal representation, 2: graphic to symbolic representation; for Test B, 1: verbal to graphic representation, 2: verbal to symbolic representation.

The similarity diagram allows for the grouping of students' responses to the tasks based on their homogeneity. Two distinct similarity groups of tasks are identified. The first group involves similarity relations among the tasks of Test A, while the second group involves similarity relations among the tasks of Test B. This finding reveals that different types of conversions among representations of the same mathematical content were approached in a completely distinct way. The starting representation of a conversion, i.e., graphic or verbal representation, seems to have influenced the students' performance, even though the tasks involved the same algebraic relations. Thus, we observe a complete separation of students' responses to the two tests even in tasks that were similar and rather "easy" for this grade of students. The similarity relations within the group of variables of the tests are also of great interest since they provide some indications of the students' way of understanding the particular algebraic relations and further support the likelihood that the phenomenon of compartmentalization was present.

For example, the similarity group of Test B is comprised of three subgroups. The **first subgroup** contains students' responses to the tasks v11b and v12b ($y<0$) and the tasks v21b and v22b ($xy>0$), that is, the two conversions from verbal to graphic representation and from verbal to symbolic representation of the first two tasks of Test B. These two tasks involve relations that represent "regions of points" and they are the easiest tasks of the test. The **second subgroup** is formed by the variables v31b ($y>x$), v41b($y=-x$), v51b ($y=3/2$) and v61b

($y = x\text{-}2$) that is the conversion from verbal to graphic representation of four relations of "functional character," as the relation of task 3 corresponds to a region of points related to the function $y=x$, while the relations of tasks 4, 5 and 6 are functions. The **third subgroup** is comprised by the variables v42b ($y=-x$), v52b ($y=3/2$) and v62b ($y=x\text{-}2$), that is, the conversion from verbal to algebraic representation of the tasks that involve functions.

To sum up, the formation of the first subgroup separately from the other two is of a "conceptual nature," since it is due to the conceptual characteristics of the relations involved, whereas, the distinction between the third subgroup and the forth subgroup is of a "representational character," since it is a consequence of the target of the conversion. To summarize, one can observe two kinds of compartmentalization in the similarity diagram: one "first order" compartmentalization (between the tasks of the two tests) and one "second order" compartmentalization (between the tasks of the same test).

The implicative diagram in Figure 2 was derived from the implicative analysis of the data and contains implicative relations, indicating whether success at a specific task implies success at another task related to the former one. The implicative relations are in line with the connections in the similarity diagram and the above remarks. In particular, one can observe the formation of two groups of implicative relations. The first group involves implicative relations among the responses to the tasks of Test B and the second group involves implicative relations among the responses to the tasks of Test A.
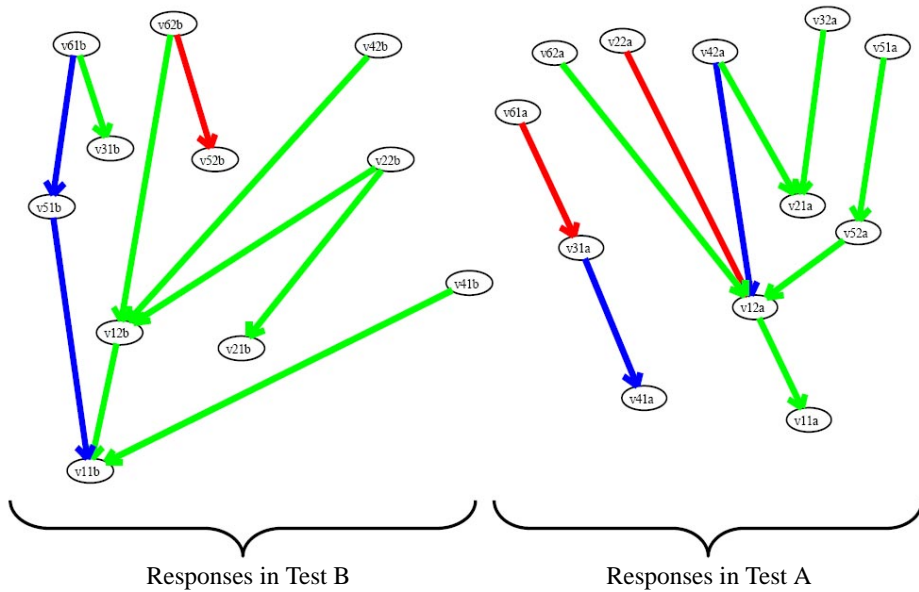
Figure 2: *Implicative diagram of 14-year-old students' responses to the tasks of Test A and Test B*

The fact that implicative relations appear only between students' responses to the tasks of the same test indicates that success at one type of conversion of an algebraic relation did not necessarily imply success at another type of conversion of the same relation. For example, students who accomplished the conversion from a graphical representation of a mathematical relation to its verbal representation were not automatically in a position to translate the same relation from its verbal representation to its graphical form successfully. This is the first order compartmentalization that appears between students' responses to the tasks of the two tests. Additionally, evidence is provided for the appearance of the second order compartmentalization, that is, between students' responses to the tasks of the same test. The implicative chain "v61a-v31a-v41a" of Test A and the implicative chain "v61b-v51b-v11b" of Test

B can be taken as examples of the second order compartmentalization, probably due to the same "target" representations of the conversions.

Other useful information can also be obtained by this implicative diagram. For example the simplest tasks in both tests are the tasks which involve the relation $y<0$ (v11), corresponding to a region of points. The students' failure in the tasks involving the particular relation (v11a or v11b) also implies failure at most of the other tasks in both tests. This inference is tenable as the implicative diagram was constructed by using the concept of "entropy." This means that for every implication where "a implies b" the counter-inverse "no a implies no b" is also valid.

Overall, based on the relations included in the similarity and the implicative diagrams for secondary school students, it can be

inferred that there was a compartmentalization between students' responses to the tasks of the first test and the tasks of the second test, which involved conversions of the same algebraic relations but different starting modes of representation (i.e., graphic and verbal respectively). Students' higher success rates at the tasks of Test B, i.e., conversions starting with graphic representations, relative to the tasks of Test A, i.e., conversions starting with verbal representations, provide further evidence for their inconsistent behaviour in the two types conversions. Another kind of compartmentalization was also uncovered within the same test, indicating students' distinct ways of carrying out conversion tasks with reference to their conceptual (kind of mathematical relation) or representational (target of the conversion) discrepancies.

4.2. Student teachers' approaches to the conversion of functions from the algebraic to the graphical representation (Study 2)

In this section, we present some elements from a study of Mousoulides and Gagatsis (2004) that used a different approach to explore the idea of the conversion between representations and the phenomenon of compartmentalization. The researchers investigated student teachers' approaches to solving tasks of functions and the connection of these approaches with complex geometric problem solving. The theoretical perspective used in their study is related to a dimension of the framework developed by Moschkovich, Schoenfeld and Arcavi (1993). According to this dimension, there are two fundamentally different perspectives from which a function is viewed, i.e., the *process perspective* and the *object perspective*. From the *process perspective,* a function is perceived of as

linking x and y values: For each value of $x,$ the function has a corresponding y value. Students who view functions under this perspective can substitute a value for x into an equation and calculate the resulting value for y or find pairs of values for x and y to draw a graph. In contrast, from the *object perspective,* a function or relation and any of its representations are thought of as entities - for example, algebraically as members of parameterized classes, or in the plane, as graphs that are thought of as being "picked up whole" and rotated or translated (Moschkovich et al., 1993). Students who view functions under this perspective can recognize that equations of lines of the form $y = 3x + b$ are parallel or can draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x,$ $g(x) = 2x + 2$) using the relationship between them *(e.g.* $g(x) = f(x) + 2$) (Knuth, 2000).

Mousoulides and Gagatsis (2004) have adopted the terms "algebraic approach" and "geometric approach" in order to emphasize the use of the algebraic expression or the graphical representations by the students in the conversion tasks and in problem solving. The algebraic approach is relatively more effective in making salient the nature of the function as a process, while the geometric approach is relatively more effective in making salient the nature of function as an object (Yerushalmy & Schwartz, 1993).

Data were obtained from 95 sophomore pre-service teachers enrolled in a basic algebra course at the University of Cyprus. A questionnaire, which consisted of four tasks and two problems, was administered at the beginning of the course. Each task involved two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical

representation. Functions in each task were related in a way such as $f(x)$, $g(x)=f(x) + c$, or $h(x)= -f(x)$, etc. The four particular tasks were as follows:

1. $y=2x$ and $y= -2x$ (T1)
2. $y= x^2$ and $y= x^2+3$ (T2)
3. $y=x^2 +3x-2$ and $y= x^2 - 3x - 2$(T3)
4. $y=x^2 +x$ and $y=x^2+2x +1$(T4)

Students were asked to sketch the graph of the second function. An example of the form in which the four tasks were proposed is as follows:

The following diagram presents a graph of the function $y=x^2 +x$. Sketch the graph of the function $y=x^2+2x +1$.
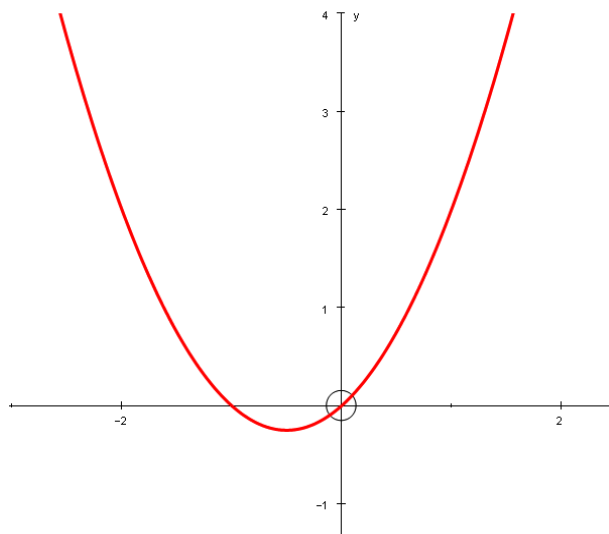

Figure 3: *The graph of the function $y=x^2+x$ (Task 4)*

It is obvious that obtaining the correct solution of the tasks did not necessarily require carrying out a treatment in the same system of representation. What was required was the conversion of the algebraic representation of a function to the graphical one, on the basis of its relation with the corresponding representations of a given function.

Additionally, students were asked to solve two problems. One of the problems consisted of textual information about a tank containing an initial amount of petrol and a tank car filling the tank with petrol. Students were asked to use the information to draw the graphs of the two linear functions, i.e. the graph of the amount of petrol in the tank with respect to time and the graph of the amount of petrol in the tank car with respect to time and to find the time at which the amounts of petrol in the tank and in the car would be equal. The other problem involved a function in a general

form $f(x) = ax^2+bx +c$. Numbers a, b and c were real numbers and the $f(x)$ was equal to 4 when $x=2$ and $f(x)$ was equal to -6 when $x=7$. Students were asked to find how many real solutions the equation $ax^2+bx +c$ had and explain their answer.

In light of the above, this study differs from the previous one in the following two basic characteristics:

• First the proposed conversions can be carried out geometrically by paying attention to the graphical representation of a given function in order to construct the representation of a second function or algebraically.

• Second, the study attempts to investigate how students' approaches to the conversions between different registers of functions are associated with their processes in problem solving on functions.

The results of this study indicated that the majority of students responded correctly in the first two tasks (T1: 73.2% and T2: 80%). Their rate of success was radically reduced in tasks involving quadratic functions involving complex transformations (T3: 41.1% and T4: 45.3%) and especially in solving complex geometric problems. More specifically, only 27.4% and 11.6% of the 95 participants provided appropriate solutions.

As regards students' approaches, more than 60% of the students that provided a correct solution followed a process perspective or the algebraic approach, which involved the construction of the function graph by finding pairs of values x and y. The other students used an object perspective or the geometric approach by observing and using the relation between the two functions. It is noteworthy that

students who chose the algebraic approach applied it even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Furthermore, in the second problem, most of the students (88.4%) failed to recognize or suggest a graphical solution as an option at all, even though the problem could not be solved algebraically.

For the similarity diagram and the implicative analysis of the data, students' answers to the tasks were codified as follow: (a) «A» was used to represent "algebraic approach – function as a process" to tasks and problems; (b) «G» stands for students who adopted a "geometric approach – function as an entity." The similarity diagram of students' responses to the tasks in Figure 4 involved two distinct clusters with reference to students' approaches. The first cluster represents the use of the algebraic approach (process perspective), while the second cluster refers to the use of the geometric approach (object perspective) and solving geometric problems. It is thus demonstrated that students who used the geometric approach in one task were likely to employ the same approach in all the other tasks. Similarly, students who used the algebraic approach employed it consistently in the tasks of the test. It can also be observed that the second cluster includes the variables corresponding to the solution of the complex geometric problems along with the variables representing the geometric approach. This means that students who effectively used the geometric approach for simple tasks on functions also succeeded in solving complex geometric problems on function. In line with the similarity diagram, success rates indicated that students who were able to use a geometric approach achieved better outcomes in solving complex function problems, probably because they were able

to observe and use the connections and the relations in the problems flexibly. The formation of the two clusters reveals that students tended to solve tasks and problems in functions using the same approach, even in tasks where a different approach was more suitable, providing support for the emergence of the phenomenon of compartmentalization in students' processes.
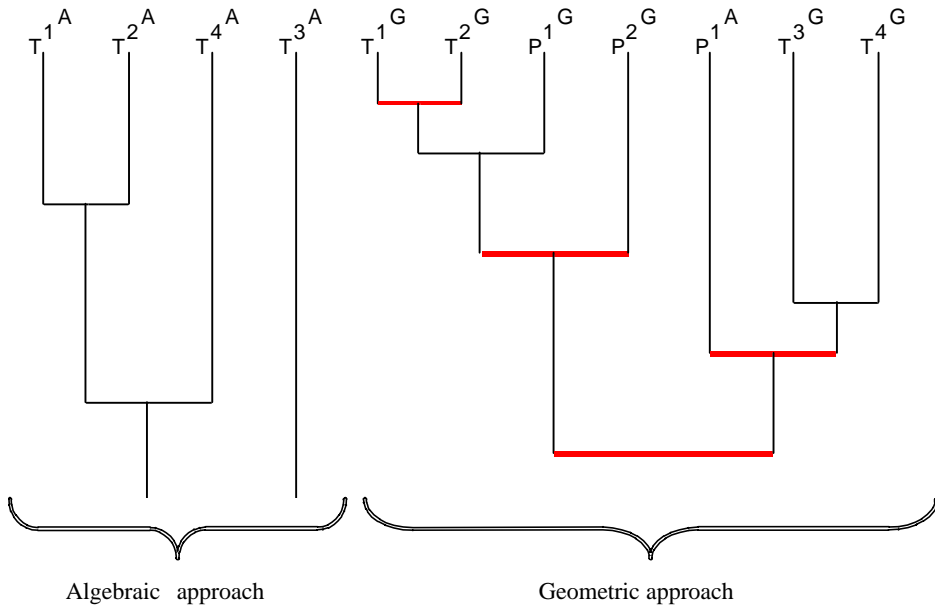

Figure 4: *Similarity diagram of student teachers' approaches to the tasks*

Note: The symbolization of the variables that were used to represent students' responses to the tasks are presented below.

1. Symbols "T1A", "T2A", "T3A" and "T4A" represent a correct algebraic approach to the tasks and "P1A" to the first problem (second problem could not be solved algebraically)

2. Symbols "T1G", "T2G", "T3G" and "T4G" represent a correct geometric approach to the tasks and "P1G" and "P2G", correct graphical solutions to the two problems

The hierarchical tree in Figure 5 involves the implicative relationships between the variables. Three groups of implicative relationships can be identified. The first group and the third group of implicative relationships include variables concerning the use of the geometric approach – object perspective and variables concerning the solution of the geometric problems. The second group involves links among variables standing for the use of the algebraic solution-process perspective. These relations are in line with the

findings derived from the similarity diagram. The establishment of these groups of links provides support once again for the consistency that characterizes students' provided solutions towards the function tasks and problems. Furthermore, the implicative relationships of the third group indicate that students who solved the second problem by applying the correct graphical solution have followed the object perspective – graphical representation for the other problem and the other two simple tasks. A possible explanation is that students who have a solid and coherent understanding of functions can recognize relations in complex geometric problems and thus can flexibly connect pairs of equations with their graphs and then easily apply the geometric approach in solving simple tasks on functions.
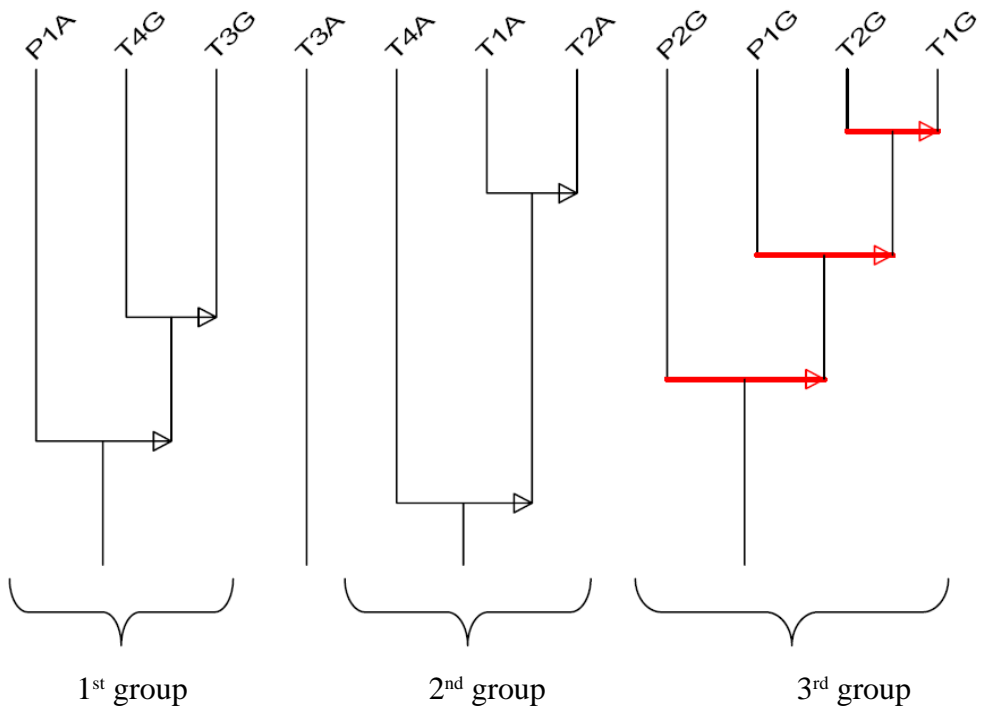

Figure 5: *Hierarchical tree illustrating implicative relations among student teachers' approaches to the tasks*

Note: The implicative relationships in bold colour are significant at a level of 99%

5. CAN WE SUCCEED AT DE-COMPARTMENTALIZATION?

Since an important aspect of this paper is to examine whether the registers of representations and the problem solving cognitive processes in functions are compartmentalized in students' thinking, we will present data from two current investigations. These studies (study 3 and study 4) are related to the previously presented studies, with their objective being to replicate previous results and support further findings for accomplishing de-compartmentalization in functions.

5.1. First effort to succeed at de-compartmentalization (Study 3)

In an attempt to accomplish de-compartmentalization, an experimental study was designed by Gagatsis, Spyrou, Evangelidou and Elia (2004). The researchers developed two experimental programs for teaching functions to university students based on two different perspectives, which are presented below. Two similar tests were administered pre- and post- the intervention in order to investigate students' understanding of functions and to compare the effectiveness of each experimental design.

One hundred fifty-seven university students participated in this study. The participants were second year students of the Department of Education (prospective teachers) who attended the course "Contemporary Mathematics" at the University of Cyprus. The students were randomly assigned to two groups which were taught by two different professors. Experimental Group 2 was comprised of 68 students and Experimental Group 2 was comprised of 81 students. The students in both groups differed in the level and length

of the mathematics courses that they had attended in school. Nevertheless, all of the students who participated in this study had received a similar curriculum on functions during the last three grades of high school.

The study was carried out in three stages. In the first stage, a pre-test was administered to both groups of students in order to investigate their initial understanding of the construct of function before the instruction. In the second stage, the two groups received instructional sessions spread over a period of the same duration for both groups. To compare the two groups, in the third stage, a post-test similar to the pre-test was used to assess students' understanding of functions.

The two experimental programs, conducted by two different university professors (Professors A and B), approached the teaching of the notion of function from two different perspectives.

Experimental Program 1 started by providing a revision of some of the functions that were already known to the students from school mathematics, physics and economics. Professor A reminded students about the difference between an equation and a function, which typically appear in a similar symbolic form. Different types of functions were presented next, starting from the simple ones and proceeding to the more complicated ones. At first, the program introduced different kinds of linear functions and described the various representations of functions in the form: $y=ax+b$. Functions with a disconnected domain were also presented. Discrete functions described by discrete types of range and the characteristic function of a set were also presented. Arrow diagrams were also introduced in order to demonstrate to the students a way to examine the ideas of one-to-one and many-to-one types of

correspondence as a condition for the definition to be held. Next, the quadratic polynomial function of the form ax^2+bx+c was taught. Special attention was given to the main features of the graph of the polynomial function (e.g., maximum and minimum points, possible roots, symmetry axis, possible qualitative manipulation of functions in the form ax^2). Various special cases and the general form of the rational function $y = \dfrac{c}{x}$ were also examined.

Trigonometric functions and their composition were studied next. The basic features and properties of the exponential functions were also discussed as well as the ill-defined functions of Weierstrass or Dirichlet without any reference to the geometrical representation. Reference was made to the inverse functions and to which functions can be inversed. The program ended by giving the set-theoretical definition of a function. The definition was then applied in order to identify whether each of the aforementioned types of relations as well as others, such as the formula of the circle, were functions or not.

Experimental Program 2 encouraged the interplay between the different modes of the representation of function in a systematic way. The instruction that was developed by Professor B on functions was based on two dimensions. The first dimension involved the intuitive approach and the definition of function. The second dimension emphasized the different representations of function. The instruction began with issues that are related to sets, the elements of a set and the operations of sets. The coordinate pairs and the Cartesian product were also discussed. The concept of correspondence was introduced, and equivalence and arrangement relations were defined. Then the activities for the study of the concept of function were based on the different relations between two sets,

namely A and B, and examples of arrow diagrams, coordinate pairs and graphs were presented.

The second dimension of the instruction concerned representations. It included the following elements: theoretical models and interesting empirical studies on the connection of representations with mathematics learning, theories on the use of semiotic representations in the teaching of mathematics and the pedagogical implications as well as the concept of function. Then the solution of tasks in graphical and algebraic representations and examples of conversion of functions from one representation to another were presented.

In the light of the above, an essential epistemological difference can be identified between the two experimental programs. Experimental Program 1 involved instruction of a classic nature, widely used at the university level. In contrast, Experimental Program 2 was based on a continuous interplay between different representations of various functions.

The pre- and the post-tests involved conversion tasks that were similar to the tasks of the test used in the study 1 described above (Gagatsis & Elia, 2005b). In addition, another two questions asked what a function is and requested two examples of functions from their application in real life situations. The tests also included tasks asking students to identify, by applying the definition of the concept, whether mathematical relations in different modes of representation (verbal expressions, graphs, arrow diagrams and algebraic expressions) were presenting functions.

Comparing the success percentages of the students before and after instruction

indicated great improvement with regards to the definition of function. In particular, while only 19% of the students gave an approximately correct definition (i.e. (i) accurate set-theoretical definition, (ii) correct reference to the relation between variables but without the definition of the domain and range, (iii) definition of a special kind of function, e.g. real, bijective, injective or continuous function) before the instruction, 69% of the students gave the corresponding definition after instruction. Students' success rates after instruction were also radically improved in most of the recognition and conversion tasks of the tests. For instance, the graph of the straight line $y=4/3$ was recognized as a function only by 26% of the students before instruction, while the graph of the line $y=-3$ was identified as a function by 82% of the students after instruction.

Analysis of the data gave four similarity diagrams. Two of the similarity diagrams involved the answers of the two experimental groups of students separately to the tasks of the test before instruction. The other two similarity diagrams included the answers of the two experimental groups of students separately, after instruction. Within the former two similarity diagrams distinct groups or subgroups of variables of students' responses in recognition tasks involving the same mode of representation of functions, i.e., in verbal form, in graphical form, in an algebraic form, in an arrow diagram, were formed separately. The particular finding revealed the consistency with which students dealt with tasks in the same representational format, but with different mathematical relations. However, lack of direct connections between variables of similar content, but different representational format, indicated that students were able to identify a function in a particular mode of representation (e.g., algebraic form), but not necessarily in another mode of representation (e.g., graphical). This inconsistent behaviour among different modes of representation was an indication of the existence of compartmentalization. This phenomenon also appeared in the similarity diagram referring to the students of Experimental Group 1, especially in the cases of the graphical representations and arrow diagrams. The compartmentalization was limited to a great extent, though, in the similarity diagram involving the responses of students of Experimental Group 2. Similarity connections were formed between students' performance in recognizing functions in different forms of representation, indicating that students dealt similarly with tasks irrespective of their mode of representation. In other words, success was independent from the mode of representation of the mathematical relation. This finding revealed that Experimental Program 2 was successful in developing students' abilities to flexibly use various modes of representations of functions and thus accomplished the breach of compartmentalization, i.e. de-compartmentalization, in their behaviour. The research in this direction, described briefly above, is still in progress.

5.2. Second effort to succeed at de-compartmentalization (Study 4)

Mousoulides and Gagatsis (2006) conducted a study exploring the effectiveness of computer based activities in de-compartmentalized registers of representations and problem solving processes in functions. A considerable number of research studies have examined the effects of technology usage on many aspects of students' mathematical achievement and attitudes, their understanding of mathematical concepts, and the instructional approaches in teaching mathematics. Despite this, only a limited number of researchers focused on the effects

of using appropriately different modes of representations and making the necessary connections between them by using technological tools (Mousoulides & Gagatsis, 2006). The investigation presented here follows the investigation presented in Section 4.2. Researchers in the aforementioned study examined whether students' work with the aid of a mathematical software package could assist students in adopting and implementing effectively the "geometric approach" to solving problems in functions and therefore promote the de-compartmentalization of registers of representations in students' thinking.

The participants were ninety sophomore students in the Department of Education. Students were attending an undergraduate course on introductory calculus. Of these, 18% were males and 82% were females. The study was conducted in three phases. In the first phase, a questionnaire similar to the one that was developed in the second study, reported here, was administered at the beginning of the course. The second phase of the study was conducted over the course of the subsequent two weeks. During this period, forty of the 90 students were randomly selected to participate in four two-hour sessions. During these sessions students, working individually or in pairs, were asked to solve problems in functions using Autograph and to present and discuss their results in discussions with the whole class. Autograph (www.autograph-math.com), a visually compelling mathematical software, was used for the purposes of the study. Autograph and other similar software packages have various features which can facilitate a constructive approach to learning mathematics (Mousoulides, Philippou & Hoyles, 2005). Autograph allows the user to "grab and move" graphs, lines and points on the screen whilst observing changes in parameters, and vice versa. Additionally, with its multiple representation capabilities, it allows the user to switch easily between numeric, symbolic and visual representations of information. A sample problem that was discussed during the second phase is presented below:

The following is the graph of the function $f(x) = ax^2+bx+c$. *Suggest possible values for* a,b,c *and explain your answers. Pose a related problem for the other students of your class that could be solved using your worksheet in Autograph.*
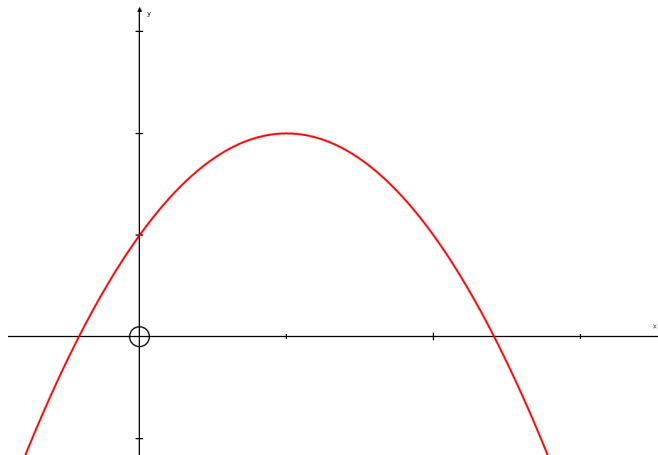

Figure 6: *The graph of the function* $f(x) = ax^2+bx+c$ *presented in one problem*

A second test, involving a second set of four tasks and two problems in functions was administered ten days after the completion of the second phase. All items in the second test were similar to the ones of the first test administered in the first phase.

Similar to the study presented in Section 4.2, researchers proposed that conversions could be carried out geometrically by focusing their attention and efforts on the relation of the symbolic representations of the two functions in order to construct the second graph or, algebraically, by selecting pairs of points to construct the new graph by "*ignoring*" its relation to the other one. Additionally, the study attempted to investigate how students' approaches in the conversions between different registers of functions were associated with their processes in problem solving. The main focus of Mousoulides and Gagatsis (2006) investigation was to examine whether student work on problems on functions with the aid of the appropriate mathematical software could result in the de-compartmentalization of the different registers of representations and their use in problem solving in functions.

The results of the study duplicate earlier findings (Mousoulides & Gagatsis, 2004), indicating that most of the students can correctly answer tasks on graphing linear (with success percentages being higher than 80%) and quadratic functions (with success percentages being higher than 65%). At the same time, their successful performance in solving related problems was limited to less than 25%. An important finding related to students' approaches showed that, in all tasks, more students preferred using the algebraic than the geometric approach. It is noteworthy that students who chose the algebraic approach applied it even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Of interest is the second problem, for which the great majority of students failed to recognize or suggest a graphical solution as an option at all, even though the problem could not be solved algebraically.

Analysis of the data from the second test showed that both groups of students improved their percentages in solving both simple tasks and problems in functions. Of interest, is the finding that students who participated in the intervention phase (Group 1) outperformed their counterparts (Group 2) in all tasks and problems. In detail, Group 1 students' percentages were higher than those of Group 2 students with percentage differences varying from 4 % to 12 % in solving tasks and from 10% to 12% in problems. Furthermore, Group 1 students significantly improved their selection of geometric approach in solving tasks and problems in functions, indicating that the exploration and discovery of open ended problems in the environment of mathematical software like Autograph might have an influence on students' selection and use of the geometric approach in functions.

The findings from the two similarity diagrams were also quite impressive. One of the similarity diagrams involved Group 2 student responses, while the second one presented the results from Group 1 students. The similarity diagram for Group 2 students involved two *distinct* clusters with reference to students' approach. In keeping with previous findings, students who used the algebraic approach employed it consistently in the tasks and problems of the test, even in cases where the use of the geometric approach was more suitable. The similarity diagram for Group 1 students showed that their responses again formed two clusters, but these clusters were not compartmentalized into algebraic and geometric approaches. Indeed, one of the

clusters showed that students were flexible in selecting the most appropriate approach for solving tasks on functions. Additionally, students were eager to switch their approach in solving a problem, especially in a problem which could not be solved using an algebraic approach. This was not the case for students in Group 2.

6. DISCUSSION

6.1. Identifying the phenomenon of compartmentalization and seeking ways to breach it

A main concern of the present paper was to investigate students' understanding of the concept of function via two perspectives. The first point of view concentrates on students' ability to handle different modes of the representation of function in tasks involving treatment and conversion and the second perspective refers to students' approaches in conversion tasks and complex function problems. Furthermore, this paper entailed some considerations with regards to the difficulties confronted by the students when dealing with different modes of mathematical representations and more specifically the phenomenon of compartmentalization. Another aim of this paper was to present two on-going investigations which attempted to design and implement different intervention programs having a common objective, i.e. to help students develop flexibility in working with various representations of function and thus accomplish de-compartmentalization of the different registers of representations in students' thinking.

The first study reported in this paper examined student performance in the conversions of algebraic relations (including functions) from one mode of representation to another. It was revealed that success in one type of conversion of an algebraic relation did not necessarily imply success in another type of conversion of the same relation. Lack of implications or connections among different types of conversion (i.e., with different starting representations or even with different target representations) of the same mathematical content indicated the difficulty in handling two or more representations in mathematical tasks. This incompetence provided a strong case for the existence of the phenomenon of compartmentalization among different registers of representation in students' thinking, even in tasks involving the same relations or functions. The differences among students' scores in the various conversions from one representation to another, referring to the same algebraic relation or function provided support for the different cognitive demands and distinctive characteristics of different modes of representation. This inconsistent behaviour was also seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion. Inconsistencies were also observed in students' responses with reference to the different conceptual features of the mathematical relations involved in the conversions, i.e. functions or not.

The most important finding of the second study was that two distinct groups were formatted with consistency, that is the algebraic and the geometric approach groups. The majority of student work with functions was restricted to the domain of the algebraic approach. This method, which is a point to point approach giving a local image of the concept of function, was followed with consistency in all of the tasks carried out by the students. Many students have not mastered even the fundamentals of the geometric approach in the domain of functions. Most of the students' understanding was limited to the use of

algebraic representations and the algebraic approach, while the use of graphical representations was fundamental in solving geometric problems. Only a few students used an object perspective and approached the function holistically, as an entity, by observing and using the association of it with the closely related function that was given. Only these students developed the ability to employ and select graphical representations, thus the geometric approach. The second study's findings are in line with the results of previous studies indicating that students cannot use the geometric approach effectively (Knuth, 2000). The fact that most of the students chose an algebraic approach (process perspective) and also demonstrated consistency in their selection of this approach, even in tasks and problems in which the geometric approach (object perspective) seemed more efficient, or the fact that they failed to suggest a graphical approach at all, is a strong indication of the phenomenon of compartmentalization in the students' processes in tasks and problems on functions involving graphical and algebraic representations.

Moreover, an important finding of the second study involved the relation between the graphical approach and geometric problem solving. This finding is consistent with the results of previous studies (Knuth, 2000; Moschkovich et al., 1993), indicating that the geometric approach enables students to manipulate functions as an entity, and thus students are capable of finding the connections and relations between the different representations involved in problems. The data presented in the second study suggested that students who had a coherent understanding of the concept of functions (geometric approach) could easily understand the relationship between symbolic and graphic representations in problems and thus were able to provide successful solutions.

In both studies presented above, the results of the statistical analysis of C.H.I.C. provided a strong case for the existence of the phenomenon of compartmentalization in students' ways of dealing with different tasks on functions. However, the findings of each of the two studies were substantial and gave different information regarding the acquisition and mastery of the concept of function. Lack of implications and similarity connections among different types of conversion of the same mathematical content in Study 1 indicated that students were not in a position to change systems of representation of the same mathematical content of functions in a coherent way. Lack of implicative and similarity connections between the geometric approach and the algebraic perspective in students' responses in Study 2 provided support for students' deficiency in flexibly employing and selecting the appropriate approach, in this case the geometric one, to sketch a graph or to solve a problem on functions. It can be asserted that registers of representations remained compartmentalized in students' minds and mathematical thinking was fragmentary and limited to the use of particular representations or a particular approach in both types of transformation, that is, treatment and conversion.

Compartmentalization, as indicated by Duval (1993; 2002) and explained empirically in the present paper, is a general phenomenon that appears not only in the learning of functions, but also in the learning of many different concepts, as pointed out at the beginning of this paper. All these findings indicate students' deficits in the coordination of different representations related to various mathematical concepts. Duval (1993; 2002) maintains that the de-compartmentalization of representations is a crucial point for the understanding of mathematical concepts.

Identifying the phenomenon of compartmentalization among the registers of representation in students' thinking on functions indicated that current instructional methods fail to help students develop a deep conceptual understanding of the particular construct. On the basis of the above findings, two current experimental efforts have been designed and carried out for the teaching of functions in order to accomplish de-compartmentalization. The former research effort (Study 3) involved two experimental programs. Experimental Program 1 involved instruction of a classic nature and one widely used at the university level. On the contrary, Experimental Program 2 was based on a continuous interplay between different representations of various functions. The other study (Study 4) involved an experimental program that promoted the exploration and discovery of open ended problems in the environment of a mathematical software program that provided multiple representation capabilities and allowed the students to switch easily between numeric, symbolic and visual representations of information. Students that participated in Experimental Program 1 of Study 3 did not show a significant improvement in the conversion tasks and continued to treat the various representations of function as distinct entities, thus demonstrating a compartmentalized way of working and thinking. As regards Experimental Program 2 of Study 3 and the experimental program of Study 4, despite their distinctive features they were both successful in stimulating a positive change in students' responses and in attaining the de-compartmentalization of representations in their performance. More specifically, the former experimental program succeeded in developing students' abilities in the conversion from one mode of representation to another. The latter program was successful in developing students' flexibility to select the most appropriate approach in solving tasks in functions and to use the geometric approach in function problems efficiently.

6.2. Recommendations for further research

Research on the identification of the phenomenon of compartmentalization in the learning of functions and other concepts should be expanded. The present paper provides support to the systematic use of appropriate statistical tools, such as the implicative statistical analysis of R. Gras (1996), to assess and analyze students' understanding of functions or other mathematical concepts. A continued research focus is needed to find ways to breach the compartmentalized way of thinking in students. The research directed towards finding ways to develop students' flexibility in using different registers of representations of functions and in moving from one to another, described briefly above, continues so as to provide explanations for the success of the two aforementioned experimental programs and to determine those features of the interventions that were particularly effective in accomplishing de-compartmentalization. There is a need for longitudinal studies in the area of registers of representations and problem solving in functions to enhance our understanding of the effectiveness and appropriateness of intervention studies like the aforementioned one. Additional studies of a qualitative nature are also needed to uncover students' difficulties in the particular domain, to expand the knowledge of how students interact with different modes of representations of functions in a conventional setting or a technological environment and how they move from a particular approach, i.e. an algebraic strategy to a more advanced one, i.e. a geometric approach in solving tasks on functions. The results of such attempts may help teachers and researchers at the

university and high school levels to place emphasis on certain dimensions of the notion of function and the pedagogical approaches to teaching functions, so that students can be assisted in constructing a solid and deep understanding of the particular concept.

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