Learning Mathematics: Increasing the Value of Initial Mathematical Wealth

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A sign may recall a certain concept or combination of concepts from somebody's memory, and can also prompt somebody to certain actions. In the first case we shall call a sign a symbol, in the second a signal. The (nature of the) effect of the sign depends on context and the actual mental situation of the reader. Van Dormolen, 1986, p.157.

RESUMEN

Usando la teoría de signos de Charles Sanders Peirce, este artículo introduce la noción de riqueza matemática. La primera sección argumenta la relación intrínseca entre las matemáticas, los aprendices de matemáticas, y los signos matemáticos. La segunda, argumenta la relación triangular entre interpretación, objetivación, y generalización. La tercera, argumenta cómo el discurso matemático es un medio potente en la objetivación semiótica. La cuarta sección argumenta cómo el discurso matemático en el salón de clase, media el aumento del valor de la riqueza matemática del alumno, en forma sincrónica y diacrónica, cuando él la invierte en la construcción de nuevos conceptos. La última sección discute cómo maestros, con diferentes perspectivas teóricas, influyen en la dirección del discurso matemático en el salón de clase y, en consecuencia, en el crecimiento de la riqueza matemática de sus estudiantes.

• **PALABRAS CLAVE:** Riqueza matemática, interpretación, relación con signos, la tríada interpretación-objetivación-generalización.

ABSTRACT

Using the Peircean semiotic perspective, the paper introduces the notion of mathematical wealth. The first section argues the intrinsic relationship between mathematics, learners of mathematics, and signs. The second argues that interpretation, objectification, and generalization are concomitant semiotic processes and that they constitute a semiotic triad. The third argues that communicating mathematically is a powerful means of semiotic objectification. The fourth section presents the notion of mathematical wealth, the learners' investment of that wealth, and the synchronic-diachronic growth of its value through classroom discourse. The last section discusses how teachers, with different theoretical perspectives, influence the direction of classroom discourse and the growth of the learner's initial mathematical wealth.

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- **KEY WORDS:** Mathematical Wealth, Interpretation, Relationships with Signs, The Triad Intepretation-Objectification-Generalization.

RESUMO

Usando a teoria de signos de Charles Sanders Peirce, este artigo introduz a noção de riqueza matemática. A primeira secção argumenta a relação intrínseca entre a matemática, os aprendizes de matemáticas, e os signos matemáticos. A segunda, argumenta a relação triangular entre interpretação, objetivação e generalização. A terceira, argumenta como o discurso matemático é um potente meio na objetivação semiótica. A quarta seção argumenta como o discurso matemático na sala de aula adequar o aumento do valor da riqueza matemática do aluno, em forma sincrônica e diacrônica, quando ele inverte a construção de novos conceitos. A última seção discute como maestros, com diferentes perspectivas teóricas, influem na direção do discurso matemático na sala de aula e, conseqüentemente, no crescimento da riqueza matemática de sus estudantes.

- **PALAVRAS CHAVES:** Riqueza matemática, interpretação, relação com signos, a tríade interpretação-objetivação-generalização.

RÉSUMÉ

En utilisant la perspective sémiotique peircienne, cet article introduit la notion de richesse mathématique. La première section soutient qu'il y a une relation intrinsèque entre les mathématiques, les apprenants des mathématiques et les signes. La deuxième section soutient que l'interprétation, l'objectivation et la généralisation sont des processus sémiotiques concomitants et qu'ils constituent une triade sémiotique. La troisième section soutient que la communication mathématique est un puissant moyen sémiotique d'objectivation. La quatrième section présente la notion de richesse mathématique, l'investissement de cette richesse par les apprenants et la croissance synchronique et diachronique de sa valeur à travers le discours de la salle de classe. La dernière section discute de la façon dont les enseignantes et enseignants, avec des perspectives théoriques différentes, agissent sur l'orientation de la discussion dans la salle de classe et sur l'enrichissement de la pensée mathématique initiale des apprenants.

- **MOTS CLÉS:** Richesse mathématique, interprétation, relation avec des signes, la triade interprétation-objectivation-généralisation.

Mathematics and its Intrisic Relationship with Signs

Since ancient times, philosophers and mathematicians alike have been concerned with the definition of mathematics as a scientific endeavor and as a way of thinking. These definitions have evolved both according to the state of the field at a particular point in time and according to different philosophical

perspectives. Davis and Hersh, assert that "each generation and each thoughtful mathematician within a generation formulates a definition according to his lights" (1981, p. 8). To define *mathematics* is as difficult as to define *signs*. It is not easy to define either one without mentioning the other, as it is not easy to define them in a paragraph and even less in a couple of sentences. Mathematicians make use of and create mathematical signs to represent, "objectify", or encode their creations. On the other hand, learners interpret mathematical signs and their relationships both to decode the conceptual objects of mathematics and to objectify (i.e., encode) their own conceptualizations.

All kinds of signs and sign systems are ubiquitous in our lives but so is mathematics. Given the fascinating and ineludible dance between mathematics and signs, it is not surprising that some mathematicians become semioticians. Peirce, for example, dedicated several volumes to analyze the relationship between mathematical objects and mathematical signs (The New Elements of Mathematics, Vols. I, II, III, IV, 1976) as well as several essays to discuss the essence of mathematics (for example, the one published in Newman's *World of Mathematics*, 1956). Peirce defines mathematics as the science that draws necessary conclusions and its propositions as "fleshless and skeletal" requiring for their interpretation an extraordinary use of abstraction. He also considers that mathematical thought is successful only when it can be generalized. Generalization, he says, is a necessary condition for mathematical thinking.

Rotman (2000), inspired by Peirce's theory, has dedicated a book to define mathematics as a sign. At the beginning of his book, he gives an overarching definition of mathematics to conclude that mathematics is essentially a symbolic practice.

> Mathematics is many things; the science of number and space; the study of pattern; an indispensable tool of technology and commerce; the methodological bedrock of the physical sciences; an endless source of recreational mind games; the ancient pursuit of absolute truth; a paradigm of logical reasoning; the most abstract of intellectual disciplines. In all of these and as a condition for their possibility, mathematics involves the creation of imaginary worlds that are intimately connected to, brought into being by, notated by, and controlled through the agency of specialized signs. One can say, therefore, that mathematics is essentially a *symbolic practice* resting on a vast and never-finished language—a perfectly correct but misleading description, since by common usage and etymology "language" is identified with speech, whereas one doesn't speak mathematics but writes it. (2000, p. ix, emphasis added).

But *where* does this symbolic practice come from? Is mathematics, as an expression of the symbolic behavior of the human species, a part of all cultures? Davis and Hersh (1981) argue that mathematics is in books, in taped lectures, in computer memories, in printed circuits, in mathematical machines, in the arrangement of the stones at Stonehenge, etc., but first and foremost, they say, it must exist first *in people's minds*. They acknowledge that there is hardly a culture, however primitive, which does not exhibit some rudimentary kind of mathematics. There seems to be a common agreement

among White (1956), Wilder (1973), Bishop (1988), and Radford (2006a) for whom mathematics is essentially a *cultural symbolic practice* that encapsulates the progressive accumulation of constructions, abstractions, generalizations, and symbolization of the human species. Progress, White contends, would have not been possible if it were not for the human ability *to give ideas an overt expression* through the use of different kinds of signs (or what he calls *the human symbolic behavior*). He asserts that human communication, as the most important and general of all symbolic behaviors, facilitates new combinations and syntheses of ideas that are passed from one individual to another and from one generation to the next. White also stresses that mathematics like language, institutions, tools, the arts, etc. is a cultural expression in the stream of the total culture. In fact, he argues that *mathematics is a synthesizing cultural process* in which concepts react upon concepts and ideas mix and fuse to form new syntheses. For White, culture is the locus of mathematical reality:

> Mathematical truths exist in the cultural tradition in which the individual is born and so they enter his mind from the outside. But apart from cultural tradition, mathematical concepts have neither existence nor meaning, and of course, cultural tradition has no existence apart from the human species. Mathematical realities thus have an existence independent of the *individual mind*, but are wholly dependent upon the *mind of the species*. (1956, pp. 2350-2351, emphasis added)

If mathematics is a symbolic practice, then the understanding of the nature of sign systems (i.e. the networking of signs over signs to create new sign-references

according to a particular syntax, grammar, and semantics) is important for the teaching and learning of mathematics. Given that individuals, by nature, possess symbolic behavior and mathematics is a symbolic practice, then why do some students come to dislike mathematics as a subject and very soon fall behind? In general, semiotics theories give us a framework to understand the mathematical and the non-mathematical behavior of our students. Among different theoretical perspectives on semiotics, Peirce's theory of signs helps us to understand how we come to construct symbolic relationships based on associative iconic and indexical ones. A relation is iconic when it makes reference to the similarity between sign and object; it is indexical when it makes reference to some physical or temporal connection between sign and object; and it is symbolic when it makes reference to some formal or merely agreed upon link between sign and object, irrespective of the physical characteristics of either sign or object.

Representation and interpretation are two important aspects of Peirce's theory. He sees *representation* as the most essential mental operation without which the notion of *sign* would make no sense (Peirce, 1903) and considers that the mind comes to associate ideas by means of referential relations between the characteristics of sign-tokens and those of the objects they come to represent. As for *interpretation*, he considers that without the interpretation of signs, communicating with the self and with others becomes an impossible task (Peirce, CP vols. 2 and 4, 1974). That is, without being interpreted, a sign as a sign does not exist. What exists is a thing or event with the potential of being interpreted and with the potential of becoming a sign. Metaphorically speaking, a sign is like a switch; it becomes relevant and its

existence becomes apparent only if it is turned on-and-off, otherwise, the switch is just a *thing* with the potential to become a switch. Likewise, a sign-token becomes a sign only when its relationship to an object or event is turned on in the flow of attention of the interpreting mind. That cognitive relationship between the sign-token and the interpreting mind is essential in Peirce's semiotic theory; in fact, it is what distinguishes his theory from other theories of signs. He crystallizes this interpreting relation between the sing-token and the individual as being the *interpretant* of the sign. This *interpretant* has the potential to generate a new sign at a higher level of interpretation and generalization. At this higher level, the new sign could, in turn, generate other iconic, indexical, or symbolic relationships with respect to the object of the sign. However, while the individual generates new *interpretants*, the object represented by the sign undergoes a transformation in the mind of the individual who is interpreting. That is, the object of the sign appears to be filtered by the continuous interpretations of the learner. In summary, Peirce considers the existence of the sign emerging both from the learner's intellectual labor to conceptualize the object of the sign and from the construction of this object in the learner's mind as a result of his intentional acts of interpretation.

> A sign stands *for* something *to* the idea that it *produces or modifies*. Or, it is a *vehicle* conveying into the mind something from without. That for which it stands is called its *object*; that which it conveys, its m*eaning*; and the idea to which it gives rise, its *interpretant*. (CP 1.339; emphasis added)

> By a Sign I mean anything whatever, real or fictile which is capable of a sensible form, is applicable to something other than itself...and that is *capable of being interpreted in another sign* which I call its Interpretant as to communicate *something that may have not been previously known about its* Object. There is thus a *triadic relation* between any Sign, and Object, and an Interpretant. (MS 654. 7) (Quoted in Pamentier, 1985; emphasis added).

Peircean semiotics helps to understand and explain many aspects of the complexity of the teaching and learning of mathematics. For example, teachers' and learners' expressions of their interpretations of mathematical signs by means of writing, reading, speaking, or gesturing; the interrelationship of the multiple representations of a concept without confounding the concept with any of its representations; and the dependency of mathematical notation on interpretation, cultural context, and historical convention. In trying to understand the semiotic nature of the teaching and the learning of mathematics, the above list about the semiotic aspects of the teaching-learning activity is anything but complete.

Brousseau, for example, contends that mathematicians and teachers both perform a "didactical practice" albeit of a different nature. Mathematicians, he says, do not communicate their results in the form in which they create them; they re-organize them, they give them the most general possible form; "they put knowledge into a communicable, decontextualized, depersonalized, detemporalized form" (1997, p. 227). This means, that they encode their creations using mathematical sign systems or they create new signs if necessary. That is, they objectify or symbolize their creations (i.e., knowledge

objects) through spacio-temporal signs. On the other hand, the teacher undertakes actions in the opposite direction. She, herself, interprets mathematical meanings embedded in spacio-temporal signs (sign-tokens), decodes conceptual objects, and looks for learning situations that could facilitate the endowment of those sign-tokens with mathematical meanings in the minds of the learners. Thus, mathematicians and teachers of mathematics have a necessary interpretative relationship with the sign systems of mathematics (i.e., semiotic mathematical systems) because they continuously use them to encode, interpret, decode, and communicate the mathematical meanings of conceptual objects.

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Teacher's and Learner's Interpretations and Objectifications

The interpretation of signs is important for two reasons. First, signs are not signs if they are not interpreted; being a sign means being a sign *of* something *to* somebody. Second, the meaning of a sign is not only in the sign but also in the mind interpreting that sign. Now the question is: Does a sign objectify? According to Peirce's definition of signs, the answer is yes. A sign does objectify (i.e., It does make tangible) the object (conceptual or material) that it *stands for*. However, the sign not only objectifies but it also communicates (to the interpreting mind) something that has not been previously known about the

object. Thus, Peirce's definition of signs implies a continuous process of interpretation and as a consequence, a concomitant process of gradual objectification.

Radford (2006b), on the other hand, considers that to objectify is to make visible and tangible something that could not be perceived before. He defines *objectification* as "an active, creative, imaginative, and interpretative social process of gradually becoming aware of mathematical objects and their properties". This definition is not in contradiction with Peirce's definition of signs. Radford (2003) also defines *means of objectification* as "tools, signs of all sorts, and artifacts that individuals intentionally use in social-meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (p. 41). This definition is also in harmony with Peirce's definition of *interpretant.*

Since mathematical objects make their presence manifest only through signs and sign systems, how can teachers help learners to enter into the world of these semiotic systems and break the code, so to speak, to "see" those objects by themselves? Which mathematical objects do learners interpret from signs[2]? Or better, what "objects" do sign-tokens stand for in the minds of learners and teachers? Would

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[2] Peirce gave several definitions of signs without contradicting previous definitions; instead he extended them. The invariant in his definitions is the triadic nature of the sign. The variation is in the names he gave to the sign-vehicle/ sign-token or material representation of the sign. First, he called *sign* the material representation of the sign, then sign-vehicle, and then representamen. Some mathematics educators have favored the sign triad object-sign-interpretant, others, like myself, have favored the sign triad object-representamen-interpretant because it does not use the word sign to indicate, at the same time, the triad and a term in the triad. In this paper, I use the words representamen, representation, and sign-token interchangeably. However, Peirce used the term representation in the general sense of being a necessary operation of the human mind.

learners and teacher 'interpret' the same mathematical objects (i.e., knowledge objects) from sign relations in mathematical sign systems? Who objectifies what? What are the "products or effects" of teacher's and learners' interpretations and objectifications? What are the teacher's interpretations of the learners' interpretations? It appears that teachers' and learners' interpretations and objectifications go hand in hand in the teaching-learning activity. Because of the triadic nature of the sign, there is a necessary and concomitant relationship between objectification and interpretation; there is no interpretation without objectification and no objectification without interpretation. In addition, these two processes are linked to a third concomitant process, the process of generalization.

Mathematicians objectify their creations inventing new mathematical signs or encoding them, using already established signs and sign systems. Teacher and learners re-create knowledge objects by interpreting mathematical signs in a variety of contexts; by doing so, they undergo their own processes of objectification. There seems to be running, in parallel, three processes of objectification: the objectification of the teacher, the objectification of the learners, and the teacher's objectification of the learners' objectifications. This seems to be a cumbersome play with words, although this is at the heart of the interrelationship between teaching and learning. Obviously, teacher and learners *objectify*, but do they objectify the same thing? Are these objectifications isomorphic or at least do they resemble each other? Is the teacher aware of these processes of objectification? If so, then the teacher has the potential: (a) to question and validate her own interpretations and objectifications; (b) to make hypotheses about the learners' objectifications; (c) to question the learners to validate her

hypothesis in order to guide their processes of interpretation and objectification; and (d) to differentiate between her interpretations and objectifications and the learners' interpretations and objectifications.

When teachers and learners engage in the teaching-learning activity, w*ho* interprets and *what* is interpreted is somewhat implied, but it is nevertheless tacit, in the processes of objectification and interpretation. Obviously, in one way or another, teachers appear to play an important role in the learners' processes of interpretation and objectification. Brousseau appears to indicate these levels of interpretation. "The teacher's work … consists of proposing a *learning situation* to the learner in such a way that [the learner] produces her knowing as a *personal answer* to a question and uses it or modifies it in order to satisfy the constraints of the *milieu* [which is managed by necessary contextual and symbolic relationships] and *not* just the *teacher's expectations*" (1997, p. 228, emphasis added). Here, Brousseau points out the difference between learners' interpretations and teachers' interpretations and intentions. The question is whether or not the teacher's intentions and interpretations are realized in the students' interpretations and objectifications. In other words, do the teacher's and the learners' interpretations and objectifications, at least, resemble each other?

The teacher may design learning situations to induce learners' *construction* of mathematical objects and relationships among those objects; or the teacher may design learning situations in which the mathematical object is directly *delivered* as if it were a *cultural artifact* ready to be "seen" and memorized by the learners, while saving them the cost of their own abstractions and generalizations. In the latter case, the learners could be

objectifying only the iconic or indexical aspects of the mathematical signs without capturing the symbolic aspects of those signs and their symbolic relations with other signs. In the former case, the learners capture both the symbolic aspects of the signs and their symbolic relations with other signs. This means that the learner is able to unfold those signs to "see" not only the symbolic aspects but also the indexical and iconic aspects embedded in them. Thus, learners and teacher could be interpreting different aspects of the mathematical signs (iconic, indexical, or symbolic) and, in consequence, interpreting the nature of mathematical objects from different levels of generalization and abstraction.

But what is the nature of the mathematical objects? How many types of objects could be interpreted from mathematical signs? Duval (2006) calls our attention to different types of *objects:*

(1) Objects as *knowledge objects* when attention is focused on the invariant of a set of phenomena or on the invariant of some multiplicity of possible representations. Mathematical objects like numbers, functions, vectors, etc. are all *knowledge objects.*

(2) Objects as *transient phenomenological objects* when the focus of attention is on this or that particular aspect of the representation given (e.g., shape, position, size, succession, etc.).

(3) Objects as *concrete objects* when the focus of attention is only on their perceptual organization.

Thus given a sign-token (i.e., a representamen or a representation), one could interpret at face value a *concrete object* if one focuses strictly on the material aspects of this semiotic means of objectification without constructing relationships with other representations. One could also interpret a *phenomenological object* if one goes beyond pure perception and focuses on aspects of those representations in space and time. Or one could also interpret a *knowledge object* if one focuses on the invariant relations in a representation or among representations. For example, Duval (2006) considers that the algebraic equation of a line and its graph could be seen as *phenomenological objects* when one focuses on the material aspects of these representations (i.e., iconic and iconic-indexical aspects of the sign-tokens or representations); they could be *knowledge objects* if one focuses on the invariance of these representations (i.e., symbolic aspects). Once one is able to interpret and to objectify *knowledge objects*, one should be able to unfold the phenomenological (i.e., iconic, iconic-indexical) and material (i.e., iconic) aspects of those objects. However, if one objectifies only *phenomenological* and *concrete objects*, one would not necessarily come up with the symbolic aspects of their corresponding *knowledge objects*.

In a nutshell, Duval's characterization of 'objects' points out the semiotic stumbling blocks of the teaching and learning of mathematics. In this characterization, the manifestation of a *knowledge object* depends not only on its representation but also on the human agency of the interpreter, user, producer, or re-producer of that object. *Objects* could be either the interpretation of pure symbolic relations embedded in the sign-tokens or representations (i.e., knowledge objects or pure signifieds); or they could be pure material tokens with no signifieds (i.e., concrete objects or concrete things); or they could be materially based tokens interpreted in time and space (i.e., phenomenological objects). The best case would be when the knowledge object is objectified in space and time with

structured signifieds and with the potential of being used again in private and inter-subjective conceptual spaces; and, vice versa, when mathematical knowledge objects are decoded from the material sign-tokens or representations without escaping their extension in space and their succession in time.

As teachers and learners engage in the teaching-learning activity, *which* objects are the teacher referring to and *which* objects are the learners interpreting, objectifying, and working with? In the best of all scenarios, teacher and learners could interpret, from the same sign-token or representation, the same knowledge object. However, sometimes learners might only be interpreting concrete objects (i.e., concrete marks) or phenomenological objects missing, in the process, the knowledge object; meanwhile the teacher might be interpreting that learners are interpreting knowledge objects. This situation would clearly mark a conceptual rupture between teacher and learners. Therefore, interpreting in the classroom is a process that unfolds at three levels: (1) the level of those who send an intentional message (the teacher or the students); (2) the level of those who receive and interpret the message (the learners or the teacher); and (3) the level of the sender's interpretation of the receiver's interpretation. Thus, in the teaching-learning activity, the interpretation process is not only a continuous process of objectification but it is also a relative process (relative not only to teachers and learners but also relative to their prior knowledge, not to mention their beliefs about knowledge and knowing).

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Communicating Mathematically as a Means of Objectification

Communication in the mathematics classroom was viewed as depending exclusively on language (syntax and grammar), the active and passive lexicon of the participants, and the nature of the content of the message (Austin and Howson, 1979). Now, we have become aware that communication depends not only on natural language but also on the specific sublanguages of different fields of study, on linguistic and non-linguistic semiotic systems, and on a variety of social and cultural contexts in which the content of the message is embedded (Halliday, 1978; Habermas, 1984; Bruner, 1986; Vygotsky, 1987; Steinbring et al. 1998). Communication is also influenced by the behavioral dispositions and expectations of the participants as well as by their intersubjective relations of power (Bourdieu, 1991). Thus, perspectives on communication, in general, appear to have gained in complexity rather than in simplicity. Hence, perspectives on communication in the mathematics classroom have changed. This communication depends on natural language, mathematical sublanguage, and mathematical sign systems that mediate teacher's and learners' interpretations of mathematical objects.

Rotman (2000) points out a special feature of mathematical communication. He contends that in order to communicate mathematically, we essentially write. He contends that writing plays not only a *descriptive* but also a *creative* role in mathematical practices. He asserts that those *things* that are described (thoughts, signifieds, and notions) and the means by which they are described (*scribbles*) make up each other in a reciprocal manner. Mathematicians, as producers of mathematics, Rotman says, think their scribbles and scribble their thinking. Therefore, one is induced to think that learners of mathematics should do the same in order to produce and increase their

personal 'mathematical wealth' as a product of their own mathematical labor. Such wealth does not accumulate all at once, but rather, it accumulates gradually in a synchronic as well as in a diachronic manner. We will enter the discussion of mathematical wealth and its synchronic-diachronic formation in the next section.

It appears that communicating mathematically is first and foremost an *act of writing* in the form of equations, diagrams, and graphs, supported all along by the specialized sublanguage of mathematics (mathematical dictionaries are a living proof that a mathematical sublanguage exists). We also need to consider that writing is not an isolated act. *Acts of writing* are concomitant with *acts of reading*, *listening*, *interpreting*, *thinking*, and *speaking*. All these acts intervene in semiotic processes of objectification resulting from personal processes of interpretation by means of contextualization and de-contextualization, concretization and generalization. That is, communicating mathematically depends on the synergy of the processes of interpretation, objectification, and generalization.

Gay (1980), Rossi-Landi (1980), and Deacon (1997) argue that any semiological system only has a finite lexicon but its semantics accounts for an unlimited series of acceptable combinations and that some of these combinations propose original ways of describing linguistic and extralinguistic reality. By the same token, the semiotic system of mathematics has a finite number of tokens and a finite set of axioms, theorems, and definitions (Ernest, 2006). When these elements are combined, they account for a large number of acceptable combinations that describe or justify, create or interpret, prove or verify, produce or decode already culturally structured mathematical objects. In

discovering, constructing, apprehending, reproducing, or creating mathematical objects, reading and writing, listening and speaking become essential means for producing and interpreting combinations of referential relations (whether iconic, indexical, or symbolic) in a space that is both visible and intersubjective.

Vygotsky (1987) contends that in any natural language the writing and speaking acts are of different nature. Writing, he says, is a monological activity in which context is mental rather than physical and therefore it does not benefit from the immediate stimulation of others. This makes writing a demanding mental activity that requires not only the syntax and grammar of the language in use, but also the conceptual objects (i.e., knowledge objects) to be encoded or decoded using particular signs or combination of signs. In contrast, Vygostsky argues that oral dialogue is characterized by the dynamics of turn-taking determining the direction of speech: in oral dialogue, questions lead to answers and puzzlements lead to explanations. Written speech, instead, is not triggered by immediate responses as in oral dialogue. In writing, the unfolding of an argument is based much more on the personal and private labor of the individual. What Vygotsky argues about written and oral speech in the context of language can be transferred to the context of mathematical communication inside and outside of the classroom. It is one thing to clarify one's mathematical ideas when debating them and another to produce them as the result of one's own isolated mental labor and personal reflection. Both types of communication are commonly used among mathematicians (Rotman, 2000). In the last decades, oral and written modes of interacting in the classroom have been accepted as appropriate ways of communicating mathematically in the

classroom (National Council of Teachers of Mathematics, Standards, 2000).

Rotman (2000) also considers that writing and thinking are interconnected and co-terminous, co-creative, and co-significant. There is no doubt that for professional mathematicians who are in the business of producing mathematical knowledge this should be the case. But are writing and thinking always interconnected, co-creative, and co-significant activities for the learners? Or are the learners using writing to take into account only the perceptual level of mathematical signs (i.e., sign-tokens or concrete objects) to automatically perform algorithmic computations in order to survive academically? Do multiple-choice exams interfere with the development of the learners' thinking-writing capacity? Do teachers make learners aware that reading, writing, listening, and speaking are effective means of objectifying mathematical knowledge objects? Do teachers make learners aware that communicating mathematically is also constituted by justifying in terms of explanation, verifications, making valid arguments, and constructing proofs?

To communicate mathematically in the classroom, the teacher has: (a) to flexibly *move* within and between different semiotic systems (e.g., ordinary language, mathematical sub-language, mathematical notations, diagrams, graphs, gestures, etc.) (Duval 2006); (b) to *refer* to mathematical objects that are other than visible and concrete (e.g., patterns, variance, and invariance across concepts) (see for example, Radford, 2003); (c) to *address* the learners in ways that are supposed to be meaningful to them (see for example, Ongstad, 2006); and (d) to *express* (verbally and nonverbally) the encoding and decoding of mathematical objects (Ongstad, 2006). Thus communicating mathematically between teacher and learners also requires the triad referring-addressing-expressing within and between several semiotic systems.

Interpreting mathematical signs is, in essence, a dynamic process of objectification in which the individual gradually becomes aware of knowledge objects represented in verbal, algebraic, visual, and sometimes imaginary representations (Davis and Hersh, 1981) and these representations have their own inherent properties. Becoming aware of knowledge objects through a variety of representations is in itself a demanding intellectual labor because of the characteristics of different representations. Skemp (1987), for example, points out differences between visual and verbal/algebraic representations: (1) Visual representations, such as diagrams, manifest a more *individual and analog* type of thinking; in contrast, verbal/algebraic representations manifest a more *socialized* type of thinking. (2) Visual representations tend to be *integrative or synthetic*; in contrast, verbal/algebraic representations are *analytical* and show detail. (3) Visual representations tend to be *simultaneous*; in contrast, verbal/algebraic representations tend to be *sequential*. (4) Visual representations tend to be *intuitive*; in contrast, verbal/algebraic representations tend to be *logical*. All these tacit differentiations are part and parcel of the tacit knowledge underpinning the classroom mathematical discourse and they may create difficulties for some learners (Presmeg, 1997). Yet another source of tacit knowledge in the classroom discourse is the variety of speech genres in mathematical discourse, for example, debating, arguing, justifying, and proving (Seeger, 1998). For Rotman, persuading, convincing, showing, and demonstrating are discursive activities with the purpose of achieving intersubjective agreement, generalization, and semiotic objectification.

This kind of *tacit knowledge* is not even remotely considered to be a part of the institutionalized school curriculum and many teachers are not even aware of it. The lack of explicitness of the tacit knowledge (expected to be understood by the learners) contributes to their abrupt and foggy entrance into the territory of the mathematical world, where those who will successfully accumulate 'mathematical wealth' are the ones who have the capacity of making *explicit* for themselves the *tacit* underpinnings of mathematical discourse and the triadic nature of the process of conceptualization (interpretation, objectification, and generalization).

To summarize, the emergence of mathematical objects and their meanings are in no way independent from intentional acts of interpretation and objectification mediated by reading and writing, speaking and listening. These acts are essential in the gradual mathematical growth of the *mathematical wealth* of the learners. Communicating mathematically in terms of reading, writing, listening, and debating should be considered *means of interpretation and objectification*. Hence, knowledge of semiotics appears to be a necessary conceptual tool in the classroom, not only for theoretical and explanatory purposes but also for pragmatic ones.

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Communicating Mathematically and Mathematical 'Wealth'

We would like to consider *mathematical wealth* as a metaphor to refer to the learner's continuous accumulation of mathematical knowledge as the product of his *intellectual labor* in an *intra-subjective* or *inter-subjective space*. This mathematical wealth is personal, although socially and culturally constituted, in addition to continuously being in the making.

As learners initiate and continue their journey in a mathematical world (which is planned by the institutionalized curriculum and/or by the learners' own interests), they continuously invest their existing mathematical wealth in order to increase its value. This investment is a continuous process of evolution, development, and transformation of the learner's referential relations using signs of iconic, indexical, and symbolic nature. Sign-tokens are not inherently icons, indices, or symbols; they are so only if interpreted in that way. The learner's interpretation of the referential relations of signs is manifested in his verbal and written responses. Say for example, that a learner is capable of keeping in memory the expression "*positive times positive is positive and negative times negative is positive*"(*). What is the meaning of this expression for a learner at different phases of his mathematical journey? Does it change? Does it remain the same?

It could be that he has memorized this expression as we memorize prayers when we are little; they just stick in our minds and we regurgitate them, even if we do not know what they mean. It could be that the learner interprets that expression as follows: "I remember that with a '-'and a '-' I can make a '+'"; and with a '+' and a '+' I can only make a '+'". In these cases, the learner has only an iconic relationship *with* the expression (*). The learner is trying to make sense by focusing on the physical resemblances of the sign-tokens. Would he be able to ascend from the level of having an iconic relation *with* the expression (*) to the level of having an indexical relation *with* it? If the learner says, for example, "I know that 2 times 3 is 6 and -2 times -3 is 6", then the learner has an iconic-indexical relation *with* the expression (*) because he has a particular case that, in a way, indicates the *possibility*

of the generality of this statement. However, when the learner comes to transform the above expression into an expression like "$xy>0$ only in cases when $x>0$ and $y>0$ or when $x<0$ and $y<0$" or to recognize that "$-x$" could be positive or negative depending on the value of x; then the learner has a symbolic relation *with* the expression (*). In the latter, the learner has come to enrich the meaning of the expression (*) as he works with variable quantities in the context of algebra.

In fact, as the learner comes to develop a symbolic relationship *with* this expression, or the expression (*) *becomes* symbolic *for* the learner, he will also come to have an iconic and iconic-indexical relationship *with* it. This is to say that once a learner has a symbolic relation *with* a sign, he would be able to unfold it into iconic and iconic-indexical relations whenever necessary. But the other way around is not necessarily true. A learner, who has an iconic or an iconic-indexical relationship *with* a sign-token (in this case the expression (*)) may not necessarily have a symbolic relationship *with* it (i.e., the sign-token does not yet stand for a knowledge object in the mind of the learner). What does this mean in terms of objects? A learner who has constructed either a concrete or a phenomenological object may very well have not yet constructed a knowledge object. However, if the learner has constructed a knowledge object, one can safely infer that he also has constructed the corresponding concrete and phenomenological objects (i.e., the learner could be able to deconstruct the knowledge object into phenomenological and concrete objects).

When a learner repeats the expression "positive times positive is positive and negative times negative is positive", it means that he could have an iconic, an iconic-indexical, or a symbolic relationship *with* the expression. What is the relationship that the learner has constructed? This is not evident until the learner has the opportunity to use it in different contextual situations. How does the teacher, who is in charge of guiding the learner, interpret the kind of relationship that the learner has *with* the expression? The teacher could have a symbolic relationship *with* the expression (*) and think that the learner also has a symbolic relationship *with* it. In addition, if the teacher considers that any sign-token or representation is inherently symbolic, independently of the learner's interpretation, she would firmly believe that the learner could have only a symbolic relationship *with* it. Henceforth, the teacher will not change her interpretation of the learner's interpretation, and this might rupture the semantic link in the communication between the teacher and the learner. The teacher's expectations would run at a level higher than the current level of the learner's possibilities. This could prompt the teacher to misjudge the capabilities of the learner and to give up on the learner instead of creating new learning situations to induce the construction of the learner's symbolic relationship *with* sign-tokens (in this case the expression (*)). The worst case would be when the learner stops increasing the value of his initial mathematical wealth and soon falls behind others and with feelings of not having any intellectual capacity for mathematics.

The teacher needs to understand that the expression (*) or any other sign could have iconic, iconic-indexical, or iconic-indexical-symbolic meanings *for* the learner at different points of his mathematical journey. That is, the teacher should be aware that what one routinely calls "symbols" are nothing else than sign-tokens that can be

interpreted at different levels of generalization. The teacher who comes to understand *what* is symbolic and for *whom, what* is iconic-indexical and for *whom, what* is iconic and for *whom,* should also come to see her teaching deeply rooted in her own learning of mathematics and in her learning of her students' learning.

A teacher unaware of hers and the learners' possible iconic, indexical, and symbolic relationships *with* signs has no grounds for making hypotheses about the learners' interpretations. Then, the teacher will only interpret her own interpretations but not those of the learners. That is, the teacher comes to collapse the three levels of interpretation (her own interpretation, the learners' interpretation, and her interpretation of the learner's interpretation) making it only one muddled level that barely reflects the cognitive reality of those involved in the teaching-learning activity. In doing so, the teacher loses cognitive contact with the learner and thus the opportunity to support his personal processes of re-organization and transformation of his prior knowledge. It is not surprising, then, that Bauersfeld (1998) noticed that learners are alone in making their own interpretations and that there is a difference between "the matter taught" and "the matter learned". In our framework, this would translate as the existence of a difference between the matter interpreted by the teacher, the matter taught by the teacher, and the matter interpreted by the learners.

At any given moment, learners start with a particular set of mathematical conceptualizations to be transformed and re-organized. This initial set of conceptual elements, with whatever mathematical value (iconic, indexical, or symbolic) , is what we would like to call the *initial*

mathematical wealth. This wealth, if invested *in* well designed learning situations using a variety of contexts, will allow the learner to embed iconic relationships into iconic-indexical relationships and to embed iconic-indexical relationships into symbolic ones. By doing so, the learner will come to construct mathematical patterns (at different levels of generalization), and regulated combinations of mathematical signs according to the structure of the mathematical sign systems he is working with at that moment. For example, learners' generalization, in the natural numbers, that multiplication makes bigger and division makes smaller, has to be re-conceptualized or re-organized when they start working with decimals. Later on, multiplication needs to be generalized as an operation with particular properties. And even later, division needs to be re-cognized and re-organized as a particular case of multiplication. That is, the learner's relationship *with* multiplication and its results needs to be transcended and attention needs to be focused on the nature of the operation itself, leaving implicit the indexicality of particular results as well as the iconicity of the sign-tokens "times" or "x" (like in 4 times 2 or $4x2$) used for multiplication in grade school. That is, multiplication, in the long run, should become a *symbolic operation* in the mind of the learner and not only the mere memorization of multiplication facts and the multiplication algorithm.

Hence, the nature of the investment of the learner's mathematical wealth resides in his capacity to produce new levels of interpretations and concomitantly new objects (concrete, phenomenological, and knowledge objects) at different levels of generality (iconic, indexical, or symbolic). This kind of investment increases the learner's mathematical wealth and goes

beyond the manipulation of "sign-tokens [3]". That is, the value of the investment increases as the learner's interpretation of signs ascends from iconic, to iconic-indexical, to iconic-indexical-symbolic along his recursive and continuous personal processes of interpretation, transformation, and re-organization. Moreover, what becomes symbolic at a particular point in time in the learner's conceptual evolution could become the iconic or iconic-indexical root of a new symbolic sign at a higher level of interpretation. For example, our middle school knowledge of the real numbers with the operations of addition and multiplication becomes the root for interpreting, later on, the field structure of real numbers (i.e., the set of real numbers with the operations of addition and multiplication constitutes an additive group and a multiplicative group respectively and also the operation of multiplication distributives over the operation of addition).

In summary, learners who become mathematically wealthy are those who, along the way, are able to *interpret knowledge objects* from concrete *sign-tokens* and, in the process, are able to transcend their phenomenological aspects (i.e., iconic-indexical) and ascend to symbolic relationships *with* them through continuous acts of interpretation, objectification, and generalization. No matter through what lens one sees teaching and learning (i.e., learners discover, construct, or apprehend mathematical concepts), this triadic intellectual process (interpretation-objectification-generalization) is in reality a continuous recursive synchronic-diachronic process in their intellectual lives. This process is not only

synchronic. It would be impossible for the learner to appreciate, all at once, current and potential meanings embedded in contextual interpretations of mathematical signs. Only when the learners have traveled the mathematical landscape for some time, they are able to "see" deeper meanings in mathematical signs as they interpret them in new contexts and in new relationships with other signs. Hence, the process is also diachronic. In the diachronicity of the process, the learner comes to understand the meaning potential of different signs.

Continuity and recurrence (i.e., going back in thought to consider something again under a new light) is the essence of this synchronic-diachronic process. Continuity and recursion allow learners (1) to carry on with their personal histories of conceptual development and evolution and (2) to transcend conceptual experiences in particular contexts through the observation of invariance and regularities as they see those experiences from new perspectives. That is, the sequential nature of the synchronic-diachronic process upholds all personal acts of interpretation, objectification, and generalization as well as of self-persuasion. Essentially, this is a mediated and a dialectical process between learner's knowing and knowledge in the permanent presence of the continuous flow of time, not only synchronically (in the short lived present) but also diachronically (across past, present, and future). As learners travel through the world of school mathematics, they construct and interpret for themselves a network of mathematical conceptualizations that is continuously re-organized through mathematical discourse

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[2] It is worthwhile to notice that the expression *manipulation of symbols* becomes an oxymoron in Peirce's theory of signs and it could be replaced by the expression *manipulation of sign-tokens*.

(reading, writing, speaking, and listening) and de-contextualized through abstraction and generalization. As the learners' networks of mathematical conceptualizations become increasingly re-organized and transformed over time, the earlier value of their *mathematical wealth* also increases.

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Where do Learners Build up and Consolidate their Mathematical Wealth?

As learners travel through a particular territory of the mathematical world (e.g., the institutionalized school curriculum) they become *mathematically wealthier* because they become better acquainted with the ins and outs of the territory (i.e., they are able to produce symbolic interpretations of signs, or they relate to signs iconically and indexically but in a systematic manner). Others have a bird's eye view of the territory (i.e., they are able to produce only isolated iconic, or indexical interpretations of signs or they relate to signs iconically or indexically but in an unsystematic manner) and soon forget they have seen the landscape because they have made no generalizations. Still others are able to finish their journey traveling on automatic mode (i.e., using calculators and memorized manipulations) to establish their own peculiar relationships *with* the mathematical code or mathematical semiotic systems. Henceforth, they are able to produce, at best, only iconic interpretations from signs that soon will be forgotten.

The learners' mathematical wealth is built in a socio-cognitive classroom environment grounded on collective mathematical discourse as opposed to the unidirectional discourse from the teacher to the students. The quality of this discourse and the teacher's focus of attention on the learners'

mathematical arguments influence the ways in which learners invest their mathematical wealth and how they become mathematically wealthier. It is well known that teachers, who are in charge of directing the classroom discourse, guide their practices according to conscious or unconscious theoretical perspectives on mathematics and the teaching of mathematics and they focus their attention on different aspects of classroom discourse. Sierpinska (1998) delineates the theoretical perspectives of teachers within three ample frameworks: *constructivist*, *socio-cultural*, and *interactionist* theories. Constructivist perspectives focus primarily on the learners' actions and speech while the actions and speech of the teacher are seen as secondary; that is, the constructivist teacher focuses essentially on the learners and their mathematics. Socio-cultural (i,e.,Vygotskian) perspectives focus on the social and historical character of human experience, the importance of intellectual labor, the mediating role of signs as mental tools, and the role of writing in the individual's intellectual development; that is, the socio-cultural teacher focuses essentially on culture and mediated socio-cognitive relations. Interactionist perspectives focus on language as a social practice; that is, the interactionist teacher focuses essentially on discourse and intersubjectivity. The *behaviorist* perspective could be added to those emphasized by Sierpinska. The behaviorist teacher focuses essentially on the learners' performance and pays little attention, if any, to the learners' ways of thinking. Finally, eclectic teachers seem to intertwine one or more theoretical frameworks according to the needs of the learners and their personal goals as teachers.

In any classroom, one needs to be cautions about what could be considered successful

classroom communication. Successful classroom discourse may not be an indication of successful mathematical communication. Steinbring et al. (1998) contend that learners may be successful in learning only the rituals of interaction with their teachers or the routine and stereotyped frames of communication (like the well-known initiation-response-evaluation and funneling patterns). This kind of communication, they argue, leaves the learners speechless mathematically although keeping the appearance of an exchange of mathematical ideas. Brousseau (1997), and Steinbring et al. (1988), among others, present us with classical examples in which teachers, consciously or unconsciously, hurry up or misguide learners' processes of interpretation. Thus, communicating mathematically is more than simple ritualistic modes of speaking or the manipulation of sign-tokens; it is based on a progressive folding of meaningful interpretations passing from iconic, to iconic-indexical, to iconic-indexical-symbolic, and vice versa the unfolding of these relations in the opposite direction. Or as Deacon (1997) puts it: "symbolic relationships are composed of indexical relationships between sets of indices, and indexical relationships are composed of iconic relationships between sets of icons" (p. 75). That is, more complex forms of objectification emerge from simpler forms (i.e., simpler forms are transcended but remain embedded in more complex ones).

This is to say that the learner's process of mathematical interpretation is mediated by mathematical sign systems (icons, indexes, or symbols and their logical and operational relations) to constitute networks of conceptualizations and strategies for meaning-making. Communicating mathematically is, in fact, a *continuous* semiotic process of *interpretation, objectification,* and *generalization.* The construction of generalizations takes the

learner from simple iconic relations, to indexical relations, and then to symbolic relation (i.e., folding of iconic relations into indexical ones, and then embedding indexical relations into symbolic ones) in order to make new interpretations and new objectifications that produce new generalizations. Moreover, deconstructing generalizations takes the learner in the opposite direction (i.e., unfolding of symbolic relations into iconic-indexical ones, and unfolding iconic-indexical relations into iconic ones) in order to exemplify, in particular cases, the skeletal invariance arrived at in generalization. Both directions are necessary because, together, they manifest not only the recursive and progressive constructive power of individual minds but also they manifest the human and socio-cultural roots of mathematical thinking.

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Concluding Remarks

Using a Peircean perspective on semiotics, this paper argues the notion of *mathematical wealth.* The initial cognitive mathematical wealth of any learner begins early in life. In his years of schooling and with the guidance of teachers, this initial wealth is progressively invested and its value gradually increased. The process of investment is, in essence, a mediated-dialectical process of decoding a variety of semiotic systems and, conversely, the encoding of thoughts and actions in those semiotic systems that intervene. Such systems could be of socio-cultural, pedagogical, or mathematical nature.

For mathematical wealth to increase in value in the process of investment, the learner has to decode not only the mathematical code but also the tacit code of socio-cognitive rules of engagement in the classroom. A priori and implicitly, he is

expected to understand, that reading and writing, constructing and interpreting mathematical arguments, listening and speaking, and justifying in the form of explanation, verification, and proof are necessary activities for the learning of mathematics. He also has to understand that these activities can effectively mediate the appropriation and construction of mathematical meanings *from* mathematical signs and the encoding of his own interpretations and meaning-making processes back *into* mathematical signs.

The paper also argues three levels of interpretation in the classroom: (a) the learner's level of mathematical interpretation; (b) the teacher's own level of mathematical interpretation; and (c) the teacher's level of interpretation of the learners' mathematical interpretations. It is also argued that mathematical meanings are not only inherent in mathematical signs but also inherent in the learner's cognitive relationship *with* those signs. Such relationships could be of an iconic, indexical, or symbolic nature. These relationships are not necessarily disconnected since an iconic relationship could ascend and become an indexical relationship, and the latter could ascend and become a symbolic relationship. Vice versa, a symbolic relationship could be unfolded into an indexical relationship, and the indexical relationship could be unfolded into an iconic relationship. In fact, when learners manipulate sign-tokens, it is sometimes necessary, for efficiency, to keep symbolic relations implicit in one's mind. Keeping the ascending and descending directions of relationships *with* signs and sign systems allow learners to move from the particular to the general and from the general to the particular. The learners' relationships *with* mathematical signs and sign systems are the result of mediated-dialectical processes between the learner's knowing and knowledge in the synchronic and diachronic triadic process of *interpretation, objectification, and generalization*. The reader is referred to Radford (2003) and Sáenz-Ludlow (2003, 2004, and 2006) for other instances of learners' processes of interpretation, objectification, and generalization.

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